

Finite Dimensional Representations of Quantum Affine Algebras

by

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Abstract

In 1987, Kirillov and Reshetikhin conjectured a formula for how certain finite-dimensional representations of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ decomposed into $U_q(\mathfrak{g})$ -modules. Their conjecture was built on techniques from mathematical physics and the fact that the characters of those particular representations seem to satisfy a certain set of polynomial relations, generalizations of the discrete Hirota equations.

We present a new interpretation of this formula, involving the geometry of weights in the Weyl chamber of \mathfrak{g} . This revision has the virtue of being computationally easy, especially compared to the original form: the Kirillov-Reshetikhin version of the formula is computationally intractable for all but the simplest cases. The original version parameterizes the pieces of the decomposition in terms of combinatorial objects called “rigged configurations” which are very hard to enumerate. We give a bijection between rigged configurations and simpler combinatorial objects which can be easily generated.

This new version of the formula also adds some structure to the decomposition: the irreducible $U_q(\mathfrak{g})$ -modules are naturally the nodes of a tree, rooted at the representation containing the original highest weight vector. This new tree structure is somewhat consistent among representations whose highest weights are different multiples of the same fundamental weight. We use this coherence of structure to calculate the asymptotics of the growth of the dimension of these representations as the multiple of the fundamental weight gets large.

We also explore further the polynomial relations that seem to hold among the characters of these representations. The fact that every finite-dimensional representation of the quantum affine algebra is a direct sum of representations of the underlying quantized Lie algebra is a very strong positivity condition. We prove that for the classical families of Lie algebras, the positivity condition and the polynomial relations leave only one choice for the characters of the quantum affine algebras — the ones predicted by the Kirillov-Reshetikhin formula. Therefore to prove the conjecture, it would suffice to verify that the characters do indeed satisfy this set of relations.

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The pictures of decompositions in Chapter 3 were made using Paul Taylor’s excellent package `diagrams.tex`. It is available from any Comprehensive T_EX Archive Network (CTAN) site as

`macros/generic/diagrams/taylor/diagrams.tex`

or directly from its home site,

`ftp://ftp.dcs.qmw.ac.uk/pub/tex/contrib/pt/diagrams/diagrams.tex`

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Chapter 1

Introduction

The theory of finite-dimensional representations of complex simple Lie algebras is well understood. Furthermore, if one Lie algebra appears as a subalgebra of another, due to a corresponding embedding of Dynkin diagrams, there are well-known branching rules for how representations of the larger algebra decompose under the action of the smaller one.

Any finite-dimensional complex simple Lie algebra \mathfrak{g} is a subalgebra of its corresponding infinite-dimensional affine Lie algebra $\hat{\mathfrak{g}}$. The quantized universal enveloping algebra of the affine Lie algebra, $U_q(\hat{\mathfrak{g}})$, is a Hopf algebra of interest to mathematicians and mathematical physicists, introduced simultaneously by Drinfeld and Jimbo around 1985. Finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$ are not well understood, and even their structure when viewed as representations of the Hopf subalgebra $U_q(\mathfrak{g})$ is not generally known.

In 1987, Kirillov and Reshetikhin conjectured a formula for how some finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$ decomposed into $U_q(\mathfrak{g})$ -modules. They looked only at representations whose highest weight is a multiple of a fundamental weight. Their conjecture was built on techniques from mathematical physics and the fact that the characters of those particular representations seem to satisfy a certain set of polynomial relations, generalizations of the discrete Hirota equations.

In Chapter 3, we give a new interpretation of this formula, involving the geometry of weights in the Weyl chamber of \mathfrak{g} . This revision has the virtue of being computationally easy, especially compared to the original form: the Kirillov-Reshetikhin version of the formula is computationally intractable for all but the simplest cases. The original version parameterized the pieces of the decomposition in terms of combinatorial objects called “rigged configurations” which are very hard to enumerate. We give a bijection between rigged configurations and simpler combinatorial objects

which can be easily generated.

This new version of the formula also adds some structure to the decomposition: the irreducible $U_q(\mathfrak{g})$ -modules are naturally the nodes of a tree, rooted at the representation containing the original highest weight vector. This new tree structure is somewhat consistent among representations whose highest weights are different multiples of the same fundamental weight. We use this coherence of structure to calculate the asymptotics of the growth of the dimension of these representations as the multiple of the fundamental weight gets large.

In Chapter 4, we explore further the polynomial relations that seem to hold among the characters of these representations. The fact that every finite-dimensional representation of the quantum affine algebra is a direct sum of representations of the underlying quantized Lie algebra is a very strong positivity condition. We prove that for the classical families of Lie algebras, the positivity condition and the polynomial relations leave only one choice for the characters of the quantum affine algebras — the ones predicted by the Kirillov-Reshetikhin formula. Therefore to prove the conjecture, it would suffice to verify that the characters do indeed satisfy this set of relations.

Chapter 5 lists some natural questions for further research. Mostly, they ask for generalizations of the notions mentioned above to other contexts, some straightforward and some completely open-ended.

Chapter 2

Overview of Yangians and Quantum Affine Algebras

2.1 The Algebras

To any finite-dimensional simple Lie algebra \mathfrak{g} , we can associate two closely-related Hopf algebras: its Yangian $Y(\mathfrak{g})$ and its quantum affine algebra $U_q(\hat{\mathfrak{g}})$.

The Yangian

The Yangian was introduced by Drinfeld in [Dr] as part of the study of solutions to the Quantum Yang-Baxter Equation (this connection is discussed in section 2.2). A second definition of Yangians in terms of generators and relations, with an easier description of the action on highest-weight modules, was given in [Dr2], and this is the one we give here.

Fix a complex simple Lie algebra \mathfrak{g} with simple roots $\alpha_1, \dots, \alpha_r$, $r = \text{rank}(\mathfrak{g})$ with respect to some chosen Cartan subalgebra. Let $C = (c_{ij})$ denote the Cartan matrix of \mathfrak{g} , and let $b_{ij} = (\alpha_i, \alpha_j)/2$ be the symmetrized version.

Definition 2.1 *The Yangian $Y(\mathfrak{g})$ is an associative algebra with generators κ_{ik} , ξ_{ik}^+ , ξ_{ik}^- , where $i = 1, \dots, r$ and $k = 0, 1, 2, \dots$, and relations*

$$[\kappa_{ik}, \kappa_{jl}] = 0, \quad [\kappa_{i0}, \xi_{jl}^\pm] = \pm 2b_{ij}\xi_{jl}^\pm, \quad [\xi_{ik}^+, \xi_{jl}^-] = \delta_{ij}\kappa_{i,k+l}, \quad (2.1)$$

$$[\kappa_{i,k+1}, \xi_{jl}^\pm] - [\kappa_{ik}, \xi_{j,l+1}^\pm] = \pm b_{ij}(\kappa_{ik}\xi_{jl}^\pm + \xi_{jl}^\pm\kappa_{ik}), \quad (2.2)$$

$$[\xi_{i,k+1}^\pm, \xi_{jl}^\pm] - [\xi_{ik}^\pm, \xi_{j,l+1}^\pm] = \pm b_{ij}(\xi_{ik}^\pm\xi_{jl}^\pm + \xi_{jl}^\pm\xi_{ik}^\pm), \quad (2.3)$$

$$\text{for } i \neq j, \quad n = 1 - a_{ij}, \quad \text{Sym}[\xi_{i,k_1}^\pm, [\xi_{i,k_2}^\pm, \dots [\xi_{i,k_n}^\pm, \xi_{jl}^\pm] \dots]] = 0 \quad (2.4)$$

where Sym is the sum over all permutations of k_1, \dots, k_n .

The action of $Y(\mathfrak{g})$ on finite-dimensional representations is similar to the situation for \mathfrak{g} itself: in any finite-dimensional module V , there is a nonzero “highest weight” vector v , unique up to multiplication by scalars, which is sent to 0 by all the ξ_{ik}^+ and which is an eigenvector for all κ_{ik} . All of V is generated by the action of the ξ_{ik}^- on v , and the analog of the Poincaré-Birkhoff-Witt theorem holds, allowing us to pick a total order on the generators such that ordered words form a linear basis for $Y(\mathfrak{g})$.

In the case of representations of \mathfrak{g} , an irreducible highest weight module is finite-dimensional if the eigenvalues of the action on the highest weight vector are nonnegative integers. There is an analog in $Y(\mathfrak{g})$. Suppose $\kappa_{ik}v = d_{ik}v$, for some $d_{ik} \in \mathbb{C}$. Then the module V is finite-dimensional if and only if

$$1 + \sum_{k=0}^{\infty} d_{ik} u^{-k-1} = \frac{P_i(u + b_{ii})}{P_i(u)}$$

where $P_i(u)$ is a polynomial in u and the left-hand side is the rational function’s Taylor series at infinity. The set of polynomials $P_1(u), \dots, P_r(u)$ are only defined up to a scalar, so we choose them to be monic. They are called the Drinfeld polynomials of V ; r -tuples of monic polynomials are in bijection with irreducible finite-dimensional $Y(\mathfrak{g})$ -modules in this way. However, given a set of Drinfeld polynomials, there is no known way to calculate a character or even the dimension of the associated representation.

Note that there is a copy of \mathfrak{g} actually embedded in $Y(\mathfrak{g})$, as the subalgebra generated by the κ_{i0} and ξ_{i0}^{\pm} . The highest weight vector of a $Y(\mathfrak{g})$ -module is therefore a highest weight vector for an action of \mathfrak{g} on the same space. The highest weights of the resulting \mathfrak{g} action on a $Y(\mathfrak{g})$ -module are exactly the degrees of the Drinfeld polynomials $P_i(u)$.

If we multiply the right-hand side of the relations (2.2) and (2.3) by h , we get defining relations for another Hopf algebra, $Y_h(\mathfrak{g})$. It is a deformation of the loop algebra of polynomial maps $\mathbb{C}^\times \rightarrow \mathfrak{g}$ with the pointwise bracket (see [ChP2] for an introduction to deformation and quantization of Hopf algebras). In the classical limit $h \rightarrow 0$, the generators κ_{ik} , ξ_{ik}^+ , ξ_{ik}^- of $Y_h(\mathfrak{g})$ are sent to the polynomial loops $H_i u^k$, $X_i u^k$ and $Y_i u^k$ in an indeterminate u . For all values of h other than $h = 0$, though, $Y_h(\mathfrak{g})$ is isomorphic; this is why we can choose to specialize to $h = 1$ and just work with $Y(\mathfrak{g})$, as above.

Quantum Affine Algebras

We now turn our attention to quantum affine algebras. These were introduced simultaneously by Drinfeld and Jimbo, also as part of the pursuit of solutions of the Quantum Yang-Baxter Equation.

The quantum affine algebra can be realized in several different ways. First, we let $\hat{\mathfrak{g}}$ denote the (untwisted) affine Kac-Moody algebra associated to the extended Dynkin diagram of \mathfrak{g} (with the added node numbered 0 and corresponding root α_0). In [Dr], Drinfeld showed that the universal enveloping algebra $U(\hat{\mathfrak{g}})$, and indeed the universal enveloping algebra of any symmetrizable Kac-Moody algebra, can be quantized to give $U_h(\hat{\mathfrak{g}})$.

The algebras $U_h(\hat{\mathfrak{g}})$ are better-understood than for arbitrary Kac-Moody algebras because they have a second realization in terms of central extension of the loop algebra $\mathfrak{g}[u, u^{-1}]$ of Laurent polynomial maps $\mathbb{C}^\times \rightarrow \mathfrak{g}$. In [Dr2] Drinfeld provided a new definition of $U_h(\hat{\mathfrak{g}})$, which we copy here, whose generators make the loop algebra structure visible: one can think of κ_{ik} , ξ_{ik}^+ and ξ_{ik}^- as $H_i u^k$, $X_i u^k$ and $Y_i u^k$.

Definition 2.2 *The quantum affine algebra $U_h(\hat{\mathfrak{g}})$ is an h -adically complete associative algebra over $\mathbb{C}[[h]]$ with generators κ_{ik} , ξ_{ik}^+ , ξ_{ik}^- and the central element c , where $i = 0, \dots, r$ and $k \in \mathbb{Z}$, and relations*

$$\begin{aligned} [c, \kappa_{ik}] &= [c, \xi_{ik}^\pm] = 0, \\ [\kappa_{ik}, \kappa_{jl}] &= 4\delta_{k,-l} k^{-1} h^{-2} \sinh(khb_{ij}) \sinh(khc/2), \\ [\kappa_{ik}, \xi_{jl}^\pm] &= \pm 2(kh)^{-1} \sinh(khb_{ij}) \exp(\mp |k|hc/4) \xi_{j,k+l}^\pm, \\ \xi_{i,k+1}^\pm \xi_{jl}^\pm - e^{\pm hb_{ij}} \xi_{jl}^\pm \xi_{i,k+1}^\pm &= e^{\pm hb_{ij}} \xi_{ik}^\pm \xi_{j,l+1}^\pm - \xi_{j,l+1}^\pm \xi_{ik}^\pm, \\ [\xi_{ik}^+, \xi_{jl}^-] &= \delta_{ij} h^{-1} \{ \psi_{i,k+l} \exp(hc(k-l)/4) - \phi_{i,k+l} \exp(hc(l-k)/4) \} \\ \text{for } i \neq j, n = 1 - a_{ij}, \text{Sym} \sum_{r=0}^n (-1)^r C_n^r (hb_{ii}/2) \xi_{ik_1}^\pm \cdots \xi_{ik_r}^\pm \xi_{jl}^\pm \xi_{ik_{r+1}}^\pm \cdots \xi_{ik_n}^\pm &= 0 \end{aligned}$$

where Sym is the sum over all permutations of k_1, \dots, k_n ,

$$C_n^r(\alpha) = \frac{\sinh n\alpha \cdot \sinh(n-1)\alpha \cdots \sinh(n-r+1)\alpha}{\sinh \alpha \cdot \sinh 2\alpha \cdots \sinh r\alpha}$$

and the ϕ_{ip} and ψ_{ip} are determined by the relations

$$\sum_p \phi_{ip} u^{-p} = \exp \left\{ -h \left(\frac{\kappa_{i0}}{2} + \sum_{p < 0} \kappa_{ip} u^{-p} \right) \right\},$$

$$\sum_p \psi_{ip} u^{-p} = \exp \left\{ h \left(\frac{\kappa_{i0}}{2} + \sum_{p>0} \kappa_{ip} u^{-p} \right) \right\}.$$

It is frequently more convenient to talk about the deformation $U_q(\hat{\mathfrak{g}})$ instead of $U_h(\hat{\mathfrak{g}})$, where $q = e^h$. Technically, $U_h(\hat{\mathfrak{g}})$ is defined over $\mathbb{C}[[h]]$, which forces us to worry about h -adic completions of algebras and requires careful thinking to specialize h to any specific nonzero value. By looking at $U_q(\hat{\mathfrak{g}})$ instead, we can deal with an algebra defined over $\mathbb{C}[q, q^{-1}]$ with q a formal variable, and can specialize q to any nonzero complex number easily. If q is a root of unity we get different behavior, corresponding to the change in $U_h(\hat{\mathfrak{g}})$ if h were nilpotent, but for generic q everything we want about $U_h(\hat{\mathfrak{g}})$ is preserved.

Equivalence of Decomposition

Finite-dimensional irreducible representations of $Y(\mathfrak{g})$ and of $U_q(\hat{\mathfrak{g}})$ are closely related. In each case they are indexed by Drinfeld polynomials P_1, \dots, P_r . As mentioned above, the degrees of the Drinfeld polynomials in the Yangian case give the highest weight of the \mathfrak{g} action on the module. Similarly, in the $U_q(\hat{\mathfrak{g}})$ context, the degrees give the highest weight of the action of $U_q(\mathfrak{g})$, which sits as a subalgebra inside of $U_q(\hat{\mathfrak{g}})$ based on the inclusion of Dynkin diagrams. In fact, it seems these two situations are identical:

Conjecture 2.3 *Let P_1, \dots, P_r be monic polynomials. Decompose the $Y(\mathfrak{g})$ module with those Drinfeld polynomials into \mathfrak{g} -modules as $\bigoplus V_\lambda^{\oplus n_\lambda}$. Likewise, decompose the $U_q(\hat{\mathfrak{g}})$ module with the same Drinfeld polynomials into $U_q(\mathfrak{g})$ -modules as $\bigoplus V_\lambda^{\oplus m_\lambda}$. Then for each highest weight λ , the multiplicities are the same: $n_\lambda = m_\lambda$.*

This appears to be a fact that everyone believes, but no one has provided a proof. Statements made in one of these two contexts have been happily transferred to the other in the literature with no comment. We regretfully continue to sweep this omission under the rug.

2.2 The Yang-Baxter Equation and the Bethe Ansatz

Yangians and quantum affine algebras originally arose in the study of mathematical physics. Most of what are now the axioms of a Hopf algebra started as ad hoc tools for finding solutions to the Quantum Yang-Baxter equation. The ties between the two fields were the main subject of the paper [Dr] in which Yangians were originally defined.

Yang-Baxter Equations

The Quantum Yang-Baxter Equation (QYBE) is the following requirement on a matrix $R \in \text{End}(V \otimes V)$:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad (2.5)$$

The equation holds in $V \otimes V \otimes V$, where R_{ij} indicates that R is acting on the i th and j th components in the tensor product. Such an R is a constant solution of the more general Quantum Yang-Baxter Equation with spectral parameters,

$$R_{12}(u-v)R_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v) \quad (2.6)$$

a functional equation for matrix-valued functions $R : \mathbb{C} \rightarrow \text{End}(V \otimes V)$

In any Hopf algebra A , there are two possible comultiplications $\Delta : A \rightarrow A \otimes A$ and its opposite $\Delta^{op} = \sigma \circ \Delta$, where σ acts on $A \otimes A$ by switching the factors. In a cocommutative Hopf algebra the two comultiplications are equal. We say A is *almost cocommutative* if they are instead conjugate; that is, if there exists an invertible element R such that $\Delta^{op} = R\Delta R^{-1}$. In many cases there is no such element R in $A \otimes A$ but there is in some completion $A \hat{\otimes} A$, which still suits our needs as long as conjugation by R stabilizes $A \otimes A$ inside $A \hat{\otimes} A$.

We further say that A is *quasitriangular* if $(\Delta \otimes \text{id})R = R_{13}R_{23}$ and $(\text{id} \otimes \Delta)R = R_{13}R_{12}$, and we call R the *universal R -matrix* of the Hopf algebra. One can easily check that the universal R matrix is automatically a solution to the QYBE.

In the case of Yangians, we get solutions to the QYBE with spectral parameters. While there is no R -matrix in $Y(\mathfrak{g}) \otimes Y(\mathfrak{g})$ itself, there is an element $R(u)$ in the completion $(Y(\mathfrak{g}) \otimes Y(\mathfrak{g}))[[u^{-1}]]$ which has the form

$$R(u) = 1 \otimes 1 + \frac{t}{u} + \sum_{n=2}^{\infty} \frac{R_n}{u^n}$$

which intertwines the comultiplications. This Taylor series is ill-suited for solving the QYBE with spectral parameters directly, since equation (2.6) would require multiplying expansions in different indeterminates. Fortunately, one can show that if we let $R(u)$ act on any finite-dimensional representation, $R(u) = f(u)R_{\text{rat}}(u)$, where $R_{\text{rat}}(u)$ is a rational function of u and $f(u)$ is meromorphic away from a countable set of points in \mathbb{C} .

Therefore finite-dimensional representations of $Y(\mathfrak{g})$ give rise to so-called rational solutions of the QYBE with spectral parameters.

Transfer matrices and the Bethe Ansatz

In the study of integrable lattice models, a central role in understanding the behavior of the system is played by a linear operator $t(u)$ acting on the space $\mathcal{H} = V_1 \otimes V_2 \otimes \cdots \otimes V_n$. This operator is called the *row-to-row transfer matrix*, and is defined as

$$t(u) = \text{tr}_0 R_{01}(u - w_1) R_{02}(u - w_2) \cdots R_{0n}(u - w_n)$$

where the factors R_{0i} act in $V_0 \otimes \mathcal{H}$ on V_0 and V_i , and the resulting operator acts in \mathcal{H} . The system is called *integrable* if $R(u)$ is a nontrivial solution of the Quantum Yang-Baxter Equation with spectral parameters (2.6). Using the QYBE, one can easily verify that $[t(u), t(v)] = 0$. In this case the transfer matrix is a generating function for the commuting quantum Hamiltonians of the associated system. The spectrum of this commuting family determines the major characteristics of the system; for an introduction to statistical mechanics and quantum integrable systems, see [ChP2].

The Bethe Ansatz is the main technique for calculating the eigenvalues of the transfer matrices, pioneered by H. Bethe in the 1930s. The eigenvalues of the transfer matrices are given as the solutions to a set of algebraic equations, the Bethe equations. The method only locates eigenvalues corresponding to Bethe vectors, eigenvectors which satisfy a certain technical condition. However, there is evidence that finding the Bethe vectors should suffice. This was the grounds for the conjectures we mention below.

Decomposition of the Tensor Product

Now consider the case $\mathcal{H} = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ where the V_i are all finite-dimensional $Y(\mathfrak{g})$ -modules, and we use the Yangian R -matrix to define the transfer matrix $t(u)$. The heart of the connection between the QYBE and representation theory is as follows:

Theorem 2.4 *View \mathcal{H} as a \mathfrak{g} -module, by letting the copy of \mathfrak{g} embedded in $Y(\mathfrak{g})$ act diagonally on the tensor product. Then the transfer matrix $t(u)$ commutes with the \mathfrak{g} action.*

Therefore every eigenspace of the transfer matrices is a sum of \mathfrak{g} -modules.

Conjecture 2.5 *The spectrum of the transfer matrix is simple with respect to the \mathfrak{g} -action.*

This is the best possible scenario. In this case the action of the transfer matrix would completely decompose the tensor product into irreducibles, and we would have a bijection between eigenvalues of $t(u)$ and highest weight vectors of \mathcal{H} . In particular, Conjecture 2.6 in the next section is precisely the statement that the multiplicity of an irreducible \mathfrak{g} -module in the tensor product is just the number of eigenvectors that are highest weight vectors with the correct highest weight.

It has been proved that the spectrum is simple in some cases. When $V_i \simeq \mathbb{C}^n$, the R -matrix comes from $Y(\mathfrak{sl}_n)$, and the spectral parameters w_1, \dots, w_n are generic, it was proved in [Ki] that the Bethe vectors lead to a representation of the correct dimension. A bijection between the Bethe vectors and the irreducible pieces of the decomposition was completed in [KKR] and [KR1]. There is considerable computational evidence, including the decompositions in section 3.5 here, that the conjecture is true in general.

2.3 A Result of Kirillov and Reshetikhin

In [KR2], Kirillov and Reshetikhin used the correspondence between irreducible \mathfrak{g} -modules in a $Y(\mathfrak{g})$ -module and solutions to the Bethe equations, along with techniques from mathematical physics, to conjecture a formula for the decomposition of certain representations of $Y(\mathfrak{g})$. Since it is sometimes unclear which statements are conjectures and which are theorems, we give a precise account of the results from that paper in this section.

First, we restrict our attention to representations of Yangians which are tensor products of $Y(\mathfrak{g})$ -modules whose highest weights (when viewed as \mathfrak{g} -modules) are multiples of a fundamental weight. Write $\alpha_1, \dots, \alpha_r$ for the fundamental roots and $\omega_1, \dots, \omega_r$ for the fundamental weights; $W_m(\ell)$ is a $Y(\mathfrak{g})$ -module with highest weight $m\omega_\ell$, for some $m \in \mathbb{Z}_+$ and $1 \leq \ell \leq r$ (see section 3.1 for precise definitions). We want to decompose

$$\bigotimes_{a=1}^N (W_{m_a}(\ell_a)|_{\mathfrak{g}}) \simeq \bigoplus_{\lambda} V_{\lambda}^{\oplus n_{\lambda}} \quad (2.7)$$

The sum runs over all weights λ less than $\sum m_a \omega_{\ell_a}$, the highest weight of the tensor product. The nonnegative integer n_{λ} is the multiplicity with which the irreducible \mathfrak{g} -module V_{λ} with highest weight λ occurs in the decomposition.

The main result of [KR2] is the following:

Conjecture 2.6 (Kirillov-Reshetikhin) Write $\lambda = \sum m_a \omega_{\ell_a} - \sum n_i \alpha_i$. Then

$$n_\lambda = \sum_{\text{partitions}} \prod_{n \geq 1} \prod_{k=1}^r \binom{P_n^{(k)}(\nu) + \nu_n^{(k)}}{\nu_n^{(k)}}$$

The sum is taken over all ways of choosing partitions $\nu^{(1)}, \dots, \nu^{(r)}$ such that $\nu^{(i)}$ is a partition of n_i which has $\nu_n^{(i)}$ parts of size n (so $n_i = \sum_{n \geq 1} n \nu_n^{(i)}$). The function P is defined by

$$\begin{aligned} P_n^{(k)}(\nu) &= \sum_{a=1}^N \min(n, m_a) \delta_{k, \ell_a} - 2 \sum_{h \geq 1} \min(n, h) \nu_h^{(k)} + \\ &\quad + \sum_{j \neq k} \sum_{h \geq 1} \min(-c_{k,j} n, -c_{j,k} h) \nu_h^{(j)} \end{aligned}$$

where $C = (c_{i,j})$ is the Cartan matrix of \mathfrak{g} , and $\binom{a}{b} = 0$ whenever $a < b$.

Earlier papers [KKR] and [KR1] gave a purely combinatorial proof of this formula in the case $\mathfrak{g} = \mathfrak{sl}_n$, where the sets of partitions which lead to nonzero binomial coefficients are called *rigged configurations*. The formula is inspired by counting solutions to the Bethe equations. These solutions form “strings” and “holes”: the numbers $\nu_n^{(k)}$ are the number of color k strings of length n , and the formula for $P_n^{(k)}(\nu)$ counts the corresponding number of holes.

While this formula is meant to apply to all complex simple Lie algebras, the remainder of the paper restricts its attention to the classical cases.

First, it is noted that the Yangian is known to act on the following spaces:

$$\begin{aligned} A_n : \quad W_1(\ell) &= V(\omega_\ell) & 1 \leq \ell \leq n \\ B_n : \quad W_1(\ell) &= V(\omega_\ell) \oplus V(\omega_{\ell-2}) \oplus V(\omega_{\ell-4}) \oplus \cdots & 1 \leq \ell \leq n-1 \\ &W_1(n) &= V(\omega_n) \\ C_n : \quad W_1(\ell) &= V(\omega_\ell) & 1 \leq \ell \leq n \\ D_n : \quad W_1(\ell) &= V(\omega_\ell) \oplus V(\omega_{\ell-2}) \oplus V(\omega_{\ell-4}) \oplus \cdots & 1 \leq \ell \leq n-2 \\ &W_1(\ell) &= V(\omega_\ell) & \ell = n-1, n \end{aligned}$$

While nontrivial to check, these decompositions are indeed the same as the ones predicted by equation (2.7) and Conjecture 2.6 in the special case that $N = 1$ and $m_1 = 1$.

Having shown the formula is true for the obvious base cases, one might hope to complete a proof of the conjecture by induction. In the case $\mathfrak{g} = \mathfrak{sl}_n$, the characters

of $W_m(\ell)$ are known to satisfy a certain set of quadratic recurrence relations. In an earlier paper ([Ki]), Kirillov showed that the characters predicted for $\mathfrak{g} = \mathfrak{sl}_n$ by Conjecture 2.6 also satisfy those same recurrence relations, using the Littlewood-Richardson rule. Since the base cases just mentioned are a complete set of “initial data” for the recurrence, this completed the proof of the conjecture in the A_n case.

The remainder of [KR2] gives a generalization of half of this proof. The authors write down a set of recurrence relations generalizing those known for A_n to the B_n , C_n and D_n cases (see section 4.2 for these and a version which covers the exceptional Lie algebras as well). Then, although the gruesome combinatorial details do not appear in the paper, they verify that the characters predicted by the conjecture obey these recurrence relations.

To prove the conjectural formulas, it only remains to show that the actual characters of the $Y(\mathfrak{g})$ modules $W_m(\ell)$ indeed satisfy these generalized recurrence relations. Unfortunately, no proof of this fact is currently known.

Practical Questions

The formula for n_λ given in Conjecture 2.6 has one major flaw: practical computation with this formula is impossible for all but the simplest examples.

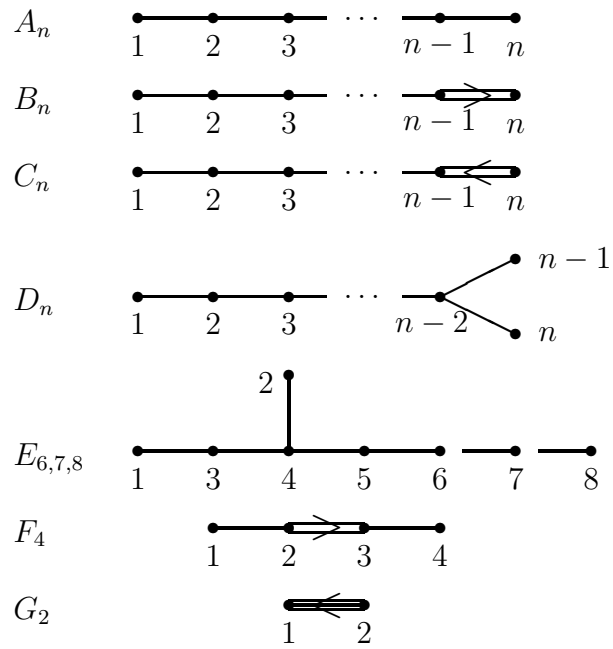
Recall that the formula is a summation over all partitions of a product of binomial coefficients. The binomial coefficient $\binom{a}{b}$ is defined to be zero whenever $a < b$, which happens any time $P_n^{(k)}(\nu)$ is negative. So the nonzero terms in the summation correspond to choices of partitions $\nu^{(1)}, \dots, \nu^{(r)}$ which have the property that $P_n^{(k)}$ is nonnegative for all $1 \leq k \leq r$ and for all $n = 1, 2, 3, \dots$

As the integers n_1, \dots, n_r in $\lambda = \sum m_a \omega_{\ell_a} - \sum n_i \alpha_i$ get larger, the total number of partitions grows much more quickly than the number which yield nonzero terms in the sum. Even in the case of the fundamental representations $W_1(\ell)$, where many of the decompositions were known using other techniques, this problem made it impossible to verify that the conjecture gave the correct results.

As a practical example, suppose one wanted to calculate the multiplicity of the trivial representation in $W_1(4)$ for E_8 (where ω_4 corresponds to the trivalent node of the Dynkin diagram). The integers n_1, \dots, n_8 are the α -coordinates of ω_4 , $(10, 15, 20, 30, 24, 18, 12, 6)$, and the number of possible choices for $\nu^{(1)}, \dots, \nu^{(r)}$ is the product of their partition numbers, 13, 339, 892, 309, 691, 024, 000.

One goal of Chapter 3 is to overcome this difficulty. Using the methods there, we find that of those 13 quintillion choices, exactly six give nonzero summands, and the total multiplicity is ten.

Figure 2.1: Numbering of nodes on Dynkin diagrams



Chapter 3

Combinatorics of Decomposition

In this chapter we investigate Kirillov and Reshetikhin's conjectured formula [KR2] for decomposing certain representations of Yangians into irreducible \mathfrak{g} -modules. We develop a practical way to compute this decomposition, and find some new structure to these modules. As a special case, we can decompose into irreducibles the tensor product of an arbitrary number of representations of \mathfrak{sl}_n whose associated Young diagrams are rectangles, in a way which is symmetric in all the factors.

A preliminary version of this chapter was published in [K]. The material has been reorganized and some changes have been made throughout. In particular, that version did not go into detail about tensor products of representations. Section 3.3 is new.

3.1 Introduction

Let \mathfrak{g} be a complex semisimple Lie algebra of rank r and $Y(\mathfrak{g})$ its Yangian, as in section 2.1. Write $\alpha_1, \dots, \alpha_r$ for the fundamental roots and $\omega_1, \dots, \omega_r$ for the fundamental weights of \mathfrak{g} . We normalize the Killing form so that the long roots have length 2.

Definition 3.1 *For $\ell = 1, 2, \dots, r$ and $m = 0, 1, 2, \dots$, let $W_m(\ell)$ denote the irreducible $Y(\mathfrak{g})$ -module with Drinfeld polynomials*

$$\begin{aligned} P_\ell(u) &= \prod_{i=1}^m \left(u + \frac{(\alpha_i, \alpha_i)}{4} (m + 1 - 2i) \right) \\ P_k(u) &= 1, \text{ for } k \neq \ell \end{aligned}$$

We allow $m = 0$, in which case $W_0(\ell)$ is the trivial representation.

Viewed as a representation of \mathfrak{g} , $W_m(\ell)$ is a (not necessarily irreducible) finite-dimensional representation in which the weight $m\omega_\ell$ occurs once and all other weights lie under $m\omega_\ell$ in the weight lattice. The Kirillov-Reshetikhin formula deals specifically with a tensor product of a number of such modules:

$$\bigotimes_{a=1}^N (W_{m_a}(\ell_a)|_{\mathfrak{g}}) \simeq \bigoplus_{\lambda} V_{\lambda}^{\oplus n_{\lambda}} \quad (3.1)$$

where V_{λ} is the irreducible \mathfrak{g} -module with highest weight λ and it occurs n_{λ} times in the decomposition of the tensor product. Let us write ω_{\max} for $\sum_{a=1}^N m_a \omega_{\ell_a}$, the highest weight (as a \mathfrak{g} -module) of the tensor product. Note that $n_{\omega_{\max}} = 1$.

As discussed in chapter 2, Kirillov and Reshetikhin used the connections with mathematical physics to arrive at the following conjecture of the multiplicities n_{λ} , where $\lambda = \omega_{\max} - \sum n_i \alpha_i$.

$$n_{\lambda} = Z(\{\ell\}, \{m\} | n_1, \dots, n_r) = \sum_{\text{partitions}} \prod_{n \geq 1} \prod_{k=1}^r \binom{P_n^{(k)}(\nu) + \nu_n^{(k)}}{\nu_n^{(k)}} \quad (3.2)$$

The sum is taken over all ways of choosing partitions $\nu^{(1)}, \dots, \nu^{(r)}$ such that $\nu^{(i)}$ is a partition of n_i which has $\nu_n^{(i)}$ parts of size n (so $n_i = \sum_{n \geq 1} n \nu_n^{(i)}$). The function P is defined by

$$\begin{aligned} P_n^{(k)}(\nu) &= \sum_{a=1}^N \min(n, m_a) \delta_{k, \ell_a} - 2 \sum_{h \geq 1} \min(n, h) \nu_h^{(k)} + \\ &\quad + \sum_{j \neq k}^r \sum_{h \geq 1} \min(-c_{kj} n, -c_{j,k} h) \nu_h^{(j)} \end{aligned} \quad (3.3)$$

where $C = (c_{ij})$ is the Cartan matrix of \mathfrak{g} . We define $\binom{a}{b}$ to be 0 whenever $a < b$. Since the values of P can be negative, many of the binomial coefficients in (3.2) can be zero.

In Section 3.2, we view the values of $P_n^{(k)}$ as the coordinates of certain strings of weights of \mathfrak{g} which lie inside the Weyl chamber. This interpretation allows us to compute the values of n_{λ} much more efficiently. Furthermore, the “initial substring” relation on the labeling by strings of weights imposes the structure of a rooted tree on the set of \mathfrak{g} -modules which make up $W_m(\ell)$.

In section 3.3, we specialize to the case where \mathfrak{g} is \mathfrak{sl}_n . Here the representations of the Yangian are irreducible when viewed as \mathfrak{g} -modules, and the conjecture has already been proven. Therefore our results give a way to compute the decomposition of a tensor product of representations of \mathfrak{sl}_n whose associated Young diagrams are all rectangles. The algorithm is symmetric in all the factors, and again imposes the structure of a rooted tree on the decomposition.

In Section 3.4, we use this new tree structure to study the asymptotics of the dimension of $W_m(\ell)$ as m gets large, based on the fact that the tree structure of $W_m(\ell)$ lifts to $W_{m+1}(\ell)$. We show that the conjecture implies that the dimension grows asymptotically to a polynomial in m , and compute the degree of this polynomial for every simply-laced \mathfrak{g} and choice of ω_ℓ .

In Section 3.5 we give a list of the decompositions of $W_m(\ell)$ for all simply-laced \mathfrak{g} and small values of m as derived numerically from the conjecture, using the results of Section 3.2. For any choice of \mathfrak{g} , representations $W_1(\ell)$ are called fundamental representations, since every finite-dimensional representation of $Y(\mathfrak{g})$ appears as a subquotient of a tensor product of such representations. The decompositions of most of the fundamental representations were calculated in [ChP] using completely different techniques, and those calculations agree with ours.

3.2 Structure in the simply-laced case

Assume that our Lie algebra \mathfrak{g} of rank r is simply-laced, and otherwise retain the setup and notation of the previous section. Then equation (3.3) simplifies to

$$P_n^{(k)}(\nu) = \sum_{a=1}^N \min(n, m_a) \delta_{k, \ell_a} - \sum_{j=1}^r c_{jk} \left(\sum_{h \geq 1} \min(n, h) \nu_h^{(j)} \right) \quad (3.4)$$

Our goal is to find all choices for $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$, where $\nu^{(i)}$ is a partition of some integer n_i , such that $P_n^{(k)}(\nu)$ is positive for all choices of k and n .

Theorem 3.2 *The pieces of the decomposition*

$$\bigotimes_{a=1}^N (W_{m_a}(\ell_a)|_{\mathfrak{g}}) \simeq \bigoplus_{\lambda} V_{\lambda}^{\oplus n_{\lambda}}$$

arise from choices of partitions $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$ which give a nonzero term in the sum in equation (3.2). Such choices are labeled by finite sequences $\mathbf{d} = (d_0, \dots, d_s)$ of weights of \mathfrak{g} , with successive differences $\delta_i = d_i - d_{i-1}$ (and $\delta_{s+1} = 0$), such that:

- (i) $d_0 = 0$ and $d_0 \prec d_1 \prec \cdots \prec d_s$,
- (ii) $\sum_{a=1}^N \min(n, m_a) \omega_{\ell_a} - d_n$ lies in the positive Weyl chamber for $0 \leq n \leq s$, and
- (iii) $\delta_i \succeq \delta_{i+1}$ for all $1 \leq i \leq s$.

where $\alpha \prec \beta$ means that $\beta - \alpha$ is in the cone of positive roots of \mathfrak{g} . If we write ω_{\max} for $\sum_{a=1}^N m_a \omega_{\ell_a}$, then the summand with label $\mathbf{d} = (d_0, \dots, d_s)$ consists of the \mathfrak{g} -module of highest weight $\omega_{\max} - d_s$ with multiplicity

$$\prod_{n \geq 1} \prod_{k=1}^r \binom{P_n^{(k)}(\mathbf{d}) + \mathbf{d}_n^{(k)}}{\mathbf{d}_n^{(k)}}$$

where the values of $P_n^{(k)}(\mathbf{d})$ and $\mathbf{d}_n^{(k)}$ are defined by the relations

$$\begin{aligned} \sum_{a=1}^N \min(n, m_a) \omega_{\ell_a} - d_n &= \sum_{k=1}^r P_n^{(k)}(\mathbf{d}) \omega_k \\ \delta_n - \delta_{n+1} &= \sum_{k=1}^r \mathbf{d}_n^{(k)} \alpha_k \end{aligned}$$

All of these multiplicities are nonzero.

Proof: Pick an arbitrary $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$, where each $\nu^{(i)}$ is a partition of some integer n_i . Then for any nonnegative integer n , the values $(P_n^{(1)}, \dots, P_n^{(r)})$ can be thought of as the ω -coordinates of some weight; define

$$\mu_n = \sum_{k=1}^r P_n^{(k)} \omega_k$$

A given ν contributes a nonzero term to the sum in (3.2) if and only if the corresponding weights $\mu_0 = 0, \mu_1, \mu_2, \dots$ all lie in the dominant Weyl chamber.

The motivation for seeing these as weights is that the sum in (3.4) can be naturally realized as subtracting some linear combination of roots. If we let

$$d_n = \sum_{k=1}^r \left(\sum_{h \geq 1} \min(n, h) \nu_h^{(k)} \right) \alpha_k \quad (3.5)$$

then $\mu_n = \sum_{a=1}^N \min(n, m_a) \omega_{\ell_a} - d_n$. Note that we have eliminated any reference to the Cartan matrix of \mathfrak{g} .

Think of $\nu^{(1)}, \dots, \nu^{(r)}$ as Young diagrams, with $\nu^{(k)}$ having $\nu_h^{(k)}$ rows of length h . The coefficient of α_k in d_n is just the number of boxes in the first n columns of $\nu^{(k)}$. Now define $\delta_i = d_i - d_{i-1}$; if we write δ_n out as a linear combination of the roots $\{\alpha_i\}$, then the α_k -coordinate is the number of boxes in the n th column of the Young diagram of $\nu^{(k)}$. Thus a sequence of vectors $d_0 = 0, d_1, d_2, \dots$ arises from partitions if and only if the δ_i are nonincreasing; that is, $\forall i \geq 1 : \delta_i \succeq \delta_{i+1}$.

If we let s be the size of the largest part in any of the partitions in ν , then $d_s = d_t$ for all $t > s$ (and s is the smallest index for which this is true). So we label each summand of the decomposition with the (strictly increasing) sequence of weights $d_0 = 0 \prec d_1 \prec \dots \prec d_s$.

We have constructed a label $\mathbf{d} = (d_0, \dots, d_s)$ for each piece in the decomposition. Conversely, from the label we can easily reconstruct the partitions, since the differences δ_i tell us the heights of successive columns of their Young diagrams. \square

We will say that the sequence $\mathbf{d} = (d_0, \dots, d_s)$ has length s , and we will call it *valid* if it satisfies conditions (i), (ii), and (iii) above. Note that the sequence of length 0 consisting of only $d_0 = 0$ is valid, arises from empty partitions, and corresponds to the $V_{\omega_{\max}}$ component of the tensor product.

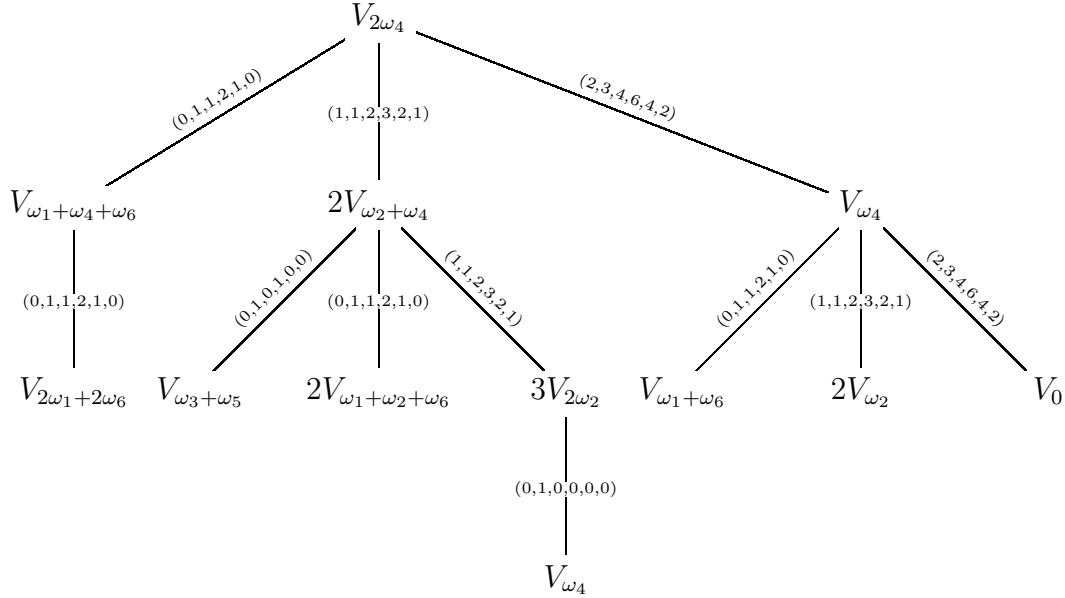
This decomposition is a refinement of the one in (3.1) since it is possible to find two different sequences d_0, \dots, d_s and d'_0, \dots, d'_t with $d_s = d'_t$. This happens any time the sum in (3.4) has more than one nonzero term. One example of this occurs in $W_2(4)$ for E_6 ; see Figure 3.1.

Corollary 3.3 *If d_0, \dots, d_s is a valid label then any initial segment $d_0, \dots, d_{s'}$ (for $0 \leq s' < s$) is a valid label also. Conversely, given any valid label d_0, \dots, d_s , we can extend it to another valid label by appending any weight d_{s+1} which satisfies the conditions that $\min(s+1, m)\omega_\ell - d_{s+1}$ is in the positive Weyl chamber and, if $s > 0$, that $d_s \prec d_{s+1} \preceq d_s + \delta_s$.*

This follows immediately from conditions (i)–(iii). This is the key result which fails to hold true when \mathfrak{g} is not simply-laced: pieces of the decomposition can still be given labels, but it is possible to have an invalid label which can be extended to a valid one. Both the tree structure and the ease of generating labels are lost. \square

Since d_0 must be 0, this completely describes an effective algorithm for computing the conjectured decomposition. The intuitive, nondeterministic version is as follows:

Algorithm 3.4 *To decompose $\bigotimes_{a=1}^N (W_{m_a}(\ell_a)|_{\mathfrak{g}})$, let $\mu_0 = 0$ and iterate the following steps for $n = 1, 2, 3, \dots$*

Figure 3.1: Tree structure of the decomposition of $W_2(4)$ for E_6 .

1. Add inc_n to μ_{n-1} , where $inc_n = \sum \omega_{\ell_a}$ for all a such that $n \leq m_a$.
2. Let μ_n be any weight in the Weyl chamber which is $\mu_{n-1} + inc_n - \delta_n$, where δ_n is any sum of positive roots less than or equal to δ_{n-1} . (If $n = 1$, ignore the δ_{n-1} part.)

Stop when $\delta_n = 0$.

The computations in Section 3.5 were computed using this algorithm.

Since truncating any label gives you another label, we can impose a tree structure on the parts of this decomposition, with a node of the tree corresponding to a summand in the decomposition from Theorem 3.2. The “children” of the node with label d_0, \dots, d_s are all the nodes indicated by Corollary 3.3; we can label the edges joining them to their parent with the various choices for the increment δ_{s+1} . For each $n \geq 0$, the n th row of the tree consists of all the nodes with labels of length n .

As an example of this structure, the tree for the decomposition of $W_2(4)$ for $\mathfrak{g} = E_6$ is given in Figure 3.1. Scalars in front of modules, as in $2V_{\omega_2+\omega_4}$, indicate multiplicity. The label (a_1, \dots, a_6) corresponds to an increment $\delta = \sum a_i \alpha_i$, so condition (iii) says that the labels along any path down from $V_{2\omega_4}$ will be nonincreasing

in each coordinate. The labels on the edges are technically unnecessary, since they can be obtained by subtracting the highest weight of the child from the highest weight of the parent. However, as we will see in section 3.4, they do record useful information that is not apparent by looking directly at the highest weights.

3.3 Example: Tensor Products of Rectangles

In this section we deal specifically with the situation when $\mathfrak{g} = \mathfrak{sl}_n$. In this and only this case, every representation of \mathfrak{g} is already a representation of $Y(\mathfrak{g})$: there is an evaluation homomorphism from $Y(\mathfrak{g})$ to the universal enveloping algebra $U(\mathfrak{g})$, which is the identity on the embedded copy $U(\mathfrak{g}) \hookrightarrow Y(\mathfrak{g})$. The results of the previous section are therefore about decomposing tensor products of representations of \mathfrak{sl}_n whose associated Young diagrams are rectangles.

In this case the decomposition formula was proved in [Ki] by counting dimensions, and an explicit bijection between rigged configurations and the Young tableaux that index pieces in the decomposition was part of the detailed papers [KKR] and [KR1] on the subject. Thus in this section we give a purely combinatorial algorithm for decomposing a tensor product of rectangles.

Note that the tensor product in equation (3.1) specializes to a tensor of single columns when the coefficients m_a are all 1. In this case the multiplicities n_λ from equation (3.2) are the Kostka numbers $K_{\lambda'\mu'}$, where λ' is the partition conjugate to λ and μ' is the partition with columns of heights $\ell_1, \ell_2, \dots, \ell_N$.

A Combinatorial Dictionary

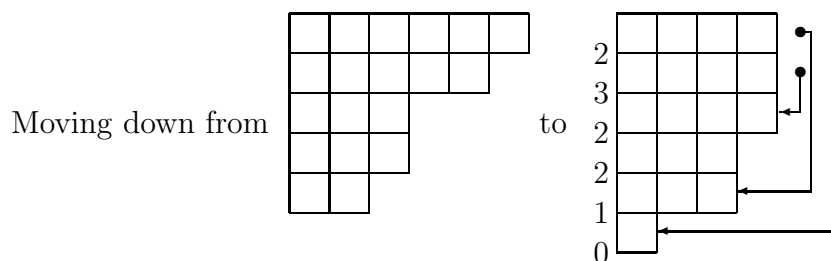
First we will fix our terminology for talking about Young diagrams, and at the same time give the correspondence between the combinatorial view and the one already presented. For convenience, we will always assume we are working with representations of \mathfrak{sl}_n for sufficiently large n ; that is, for n large enough that we never refer to ω_k for $k > n$. (As a result, we could be working in \mathfrak{gl}_n just as easily as in \mathfrak{sl}_n .)

The representation V_λ corresponds to a Young diagram, with each fundamental weight ω_k corresponding to a column of height k . That is, if we write $\lambda = \sum_{k=1}^n b_k \omega_k$, the Young diagram has b_k columns of height k . The representations $W_m(\ell)$ in this case are just $V_{m\omega_\ell}$, corresponding to a rectangle with m columns and ℓ rows.

Given the Young diagram for λ , we get the diagram for $\lambda - \alpha_i$ by moving one box down a row, from row i to row $i + 1$. Row $i + 1$ must be shorter than row

i , or else $\lambda - \alpha_i$ is not inside the Weyl chamber. The \preceq relation of the previous section therefore translates into the *dominance* ordering on Young diagrams: we say Y_1 dominates Y_2 if we can obtain Y_2 from Y_1 by moving some boxes to lower rows of the diagram. Equivalently, Y_1 dominates Y_2 if the number of boxes in the top k rows of Y_1 is at least the number in the top k rows of Y_2 , for every k .

The notion of moving down in the Weyl chamber is important enough that we will invent a notation for recording how far down we have moved in the dominance ordering. We will record how many boxes move past the line which is the bottom of the k th row; call these the *dominance numbers* of the move. We write the numbers in a column along the left side of the diagram. For example,



corresponds to subtracting $2\alpha_1 + 3\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$ from the weight $\omega_1 + 2\omega_2 + \omega_4 + 2\omega_5$. The dominance numbers of the move are 2, 3, 2, 2, 1.

We can now rephrase Algorithm 3.4 in terms of operations on Young diagrams. Figure 3.2 illustrates this process for a simple example.

Algorithm 3.5 *Every representation that occurs in the decomposition of the tensor product of representations of \mathfrak{sl}_n corresponding to rectangular diagrams R_1, R_2, \dots, R_N can be found as follows. Let $Y_0 = \emptyset$, the empty Young diagram, and iterate the following steps for $n = 1, 2, 3, \dots$*

1. *Remove the first column from each of R_1, R_2, \dots, R_N . Add those columns to Y_{n-1} to form a new diagram, Y'_n .*
2. *Let Y_n be any diagram which can be obtained from Y'_n by moving some boxes of Y'_n down to lower rows, such that the dominance numbers of Y_n are each less than or equal to the corresponding dominance number for Y'_{n-1} . (Any numbers are allowed if $n = 1$).*

Stop at any point. (Equivalently, choose to not move down at all on the next iteration, so all the dominance numbers are 0.) There is a piece of the decomposition corresponding to the final Y_n with all remaining columns of R_1, R_2, \dots, R_N added in.

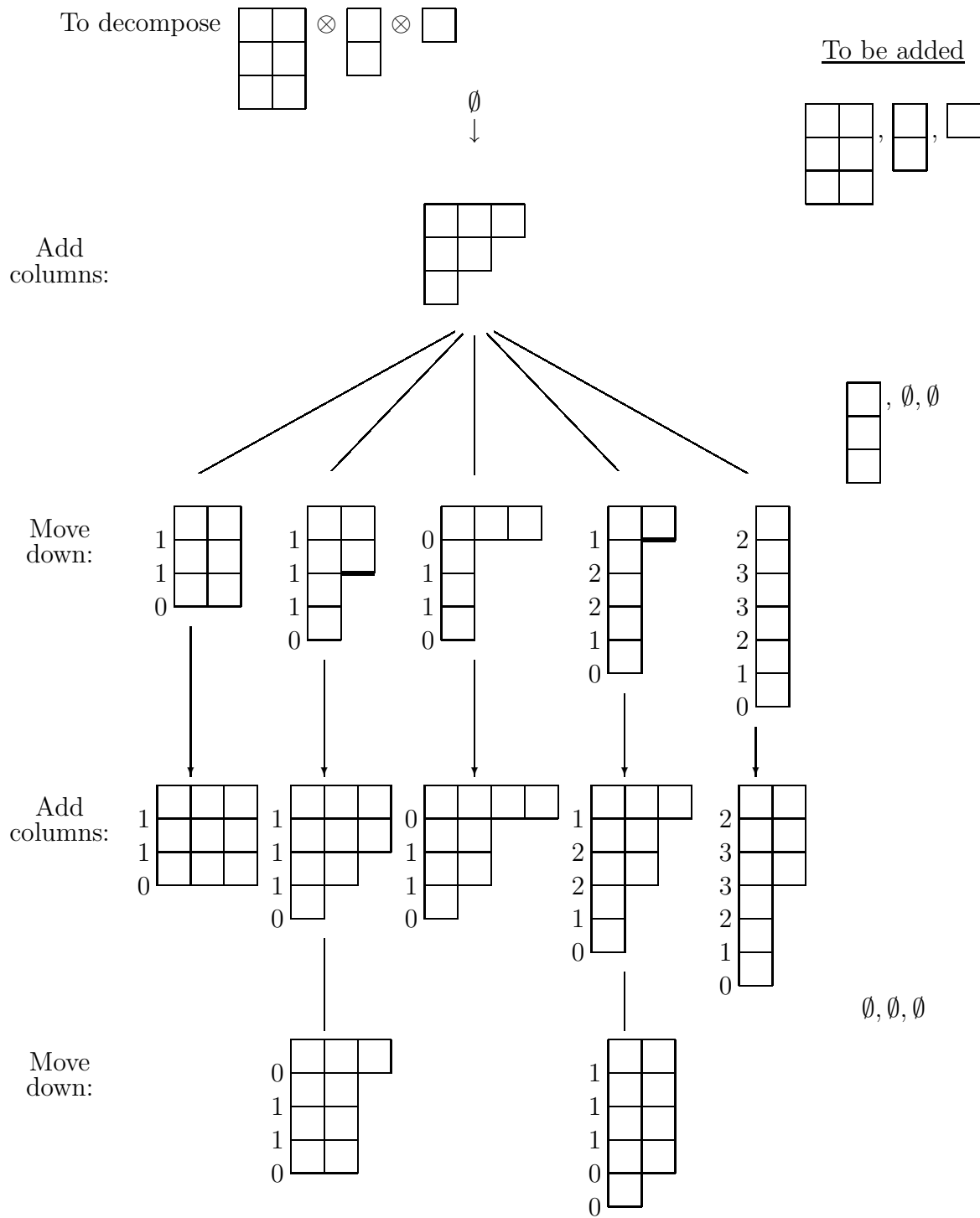
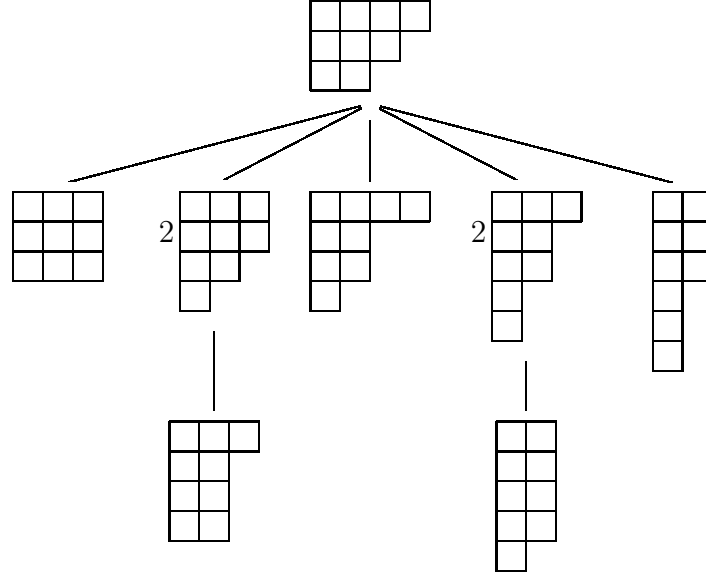


Figure 3.2: Steps in decomposing a tensor of rectangles.

Figure 3.3: Decomposition of $V_{2\omega_3} \otimes V_{\omega_2} \otimes V_{\omega_1}$ (with multiplicities).

Finally, we need to address the multiplicities given by the binomial coefficients in Theorem 3.2. Suppose we carry out Algorithm 3.5 and choose diagrams Y_1, Y_2, \dots, Y_s and then stop. The associated multiplicity is

$$\prod_{n=1}^s \prod_{k=1}^r \binom{P_n^{(k)} + \mathbf{d}_n^{(k)}}{\mathbf{d}_n^{(k)}}$$

where this time the values of P and \mathbf{d} are given by

$$\begin{aligned} P_n^{(k)} &= \text{number of columns of height } k \text{ in } Y_n, \\ \mathbf{d}_n^{(k)} &= \text{decrease in } k\text{th completion number from } Y_n \text{ to } Y_{n+1} \end{aligned}$$

We consider all the completion numbers of Y_{s+1} to be 0.

Graphically, this means we get a contribution to the multiplicity when we move from Y_n to Y_{n+1} if the k th completion number decreases and the k th row of Y_n overhangs the $k+1$ st row. If $n \neq s$ then the multiplicities of the entire section of the tree from Y_{k+1} down have that binomial coefficient as a factor. Unfortunately, this never happens in Figure 3.2; the smallest examples in which this comes up are too large to include here. (The interested reader can see it take place by trying the example in Figure 3.2 with $2\omega_2$ instead of ω_2 .)

If $n = s$ then we are looking for an instance where the k th row of Y_n overhangs the $k + 1$ st next to any completion number which is nonzero; in this case the factor affects the multiplicity of only the Y_s node on the tree, not those below it. This happens twice in Figure 3.2; the overhangings are marked in bold. The complete decomposition, including multiplicities, is presented in Figure 3.3.

3.4 The Growth of Trees

In this section we stop talking about tensor products and just look at decompositions of the modules $W_m(\ell)$ themselves. We show that the trees of the decompositions of $W_m(\ell)$ for different values of m are compatible with one another. As an application of this newfound structure, we will prove that as m gets large, the dimension of the representation $W_m(\ell)$ grows like a polynomial in m , and will give a method to compute the degree of the polynomial growth. All statements assume the conjectural formulas for multiplicities of \mathfrak{g} -modules. Roots and weights are numbered as in Table 2.1 (p. 12).

We begin with another corollary of Theorem 3.2, whose notation we retain.

Corollary 3.6 *If d_0, \dots, d_s is a valid label for a piece of the decomposition of $W_m(\ell)$, then it is also a valid label for $W_{m'}(\ell)$ for any $m' > m$, and for any $m' \geq s$.*

Both parts are based on the fact that condition (ii) of Theorem 3.2 is the only one that depends on m . For $m' > m$, if $\min(m, n)\omega_\ell - d_n$ is a nonnegative linear combination of the $\{\omega_\ell\}$ then adding some nonnegative multiple of ω_ℓ will not change that fact. For $m' \geq s$, the value of m' is irrelevant; the weights we look at are just $n\omega_\ell - d_n$ for $0 \leq n \leq s$. \square

If we can lift labels from $W_m(\ell)$ to $W_{m+1}(\ell)$, we can also lift the entire tree structure. Specifically, the lifting of labels extends to a map from the tree of $W_m(\ell)$ to the tree of $W_{m+1}(\ell)$ which preserves the increment δ of each edge and lifts each V_λ to $V_{\lambda+\omega_\ell}$. The $m' \geq s$ part of Corollary 3.6 tells us that this map is a bijection on rows $0, 1, \dots, m$ of the trees, where the labels have length $s \leq m$. On this part of the tree, multiplicities are also preserved. This follows from the formula for multiplicities in Theorem 3.2: the only values of $P_n^{(k)}(\mathbf{d})$ that change are for $n = m + 1$, but $\mathbf{d}_n^{(k)} = 0$ when n is greater than the length of the label, so the product of binomial coefficients is unchanged.

Definition 3.7 *Let $T(\ell)$ be the tree whose top n rows are those of $W_m(\ell)$ for all $m \geq n$.*

The highest weight associated with an individual node appearing in $T(\ell)$ is only well-defined up to addition of any multiple of ω_ℓ , but the difference δ between any node and its parent is well-defined. (These differences are the labels on the edges of the tree in Figure 3.1.) We can characterize each node by the string of successive differences $\delta_1 \succeq \delta_2 \succeq \cdots \succeq \delta_s$ which label the s edges in the path from the root of the tree to that node. The multiplicity of a node of $T(\ell)$ is well-defined, as already noted.

Claim: *For a fixed ℓ , the dimension of $W_m(\ell)$ grows as a polynomial in m , whose degree we can calculate.*

We will study the growth of the tree $T(\ell)$. The precise statement of the claim is in Theorem 3.10.

The tree of $W_m(\ell)$ matches $T(\ell)$ exactly in the top m rows. The number of rows in the tree of $W_m(\ell)$ is bounded by the largest α -coordinate of $m\omega_\ell$, since if $\delta_1, \dots, \delta_s$ is a label of $W_m(\ell)$ then $m\omega_\ell - \sum_{i=1}^s \delta_i$ must be in the positive Weyl chamber, and whatever α -coordinate is nonzero in δ_s must be nonzero in all of the δ_i . Therefore to prove that the dimension of $W_m(\ell)$ grows as a polynomial in m , it suffices to prove that the dimension of the part of $W_m(\ell)$ which corresponds to the top m rows of $T(\ell)$ does so.

Now we need to examine the structure of the tree $T(\ell)$. The path $\delta_1, \dots, \delta_s$ to reach a vertex is a sequence of weights whose α -coordinates are nonincreasing. Write this instead as $\Delta_1^{m_1} \dots \Delta_t^{m_t}$ where the Δ_i are strictly decreasing and m_i is the number of times Δ_i occurs among $\delta_1, \dots, \delta_s$; we will say this path has *path-type* $\Delta_1 \dots \Delta_t$. The number of path-types that can possibly appear in the tree $T(\ell)$ is finite, since each Δ_i is between ω_ℓ and 0 and has integer α -coordinates.

We need to understand which path-types $\Delta_1 \dots \Delta_t$ and which choices of exponents m_i correspond to paths which actually appear in $T(\ell)$. Given a path $\delta_1, \dots, \delta_s$, assume that $m > s$ and recall $\mu_n = n\omega_\ell - d_n = n\omega_\ell - \sum_{i=1}^n \delta_i$. Condition (ii) from Theorem 3.2 requires that μ_n is in the positive Weyl chamber for $1 \leq n \leq s$; that is, the ω -coordinates of μ_n must always be nonnegative. (These coordinates are just the values of $P_n^{(k)}$ from Theorem 3.2.) Since $\mu_n = \mu_{n-1} + \omega_\ell - \delta_n$, we need to keep track of which ω -coordinates of $\omega_\ell - \delta_n$ are positive and which are negative.

Definition 3.8 *For a path-type $\Delta_1 \dots \Delta_t$, we say that Δ_i provides ω_k if the ω_k -coordinate of $\omega_\ell - \Delta_i$ is positive, and that it requires ω_k if the coordinate is negative.*

Geometrically, Δ_i providing ω_k means that each Δ_i in the path moves the sequence of μ s away from the ω_k -wall of the Weyl chamber, while requiring ω_k moves

towards that wall. The terminology is justified by restating what condition (ii) implies about path-types in these terms:

Lemma 3.9 *The tree $T(\ell)$ contains paths of type $\Delta_1 \dots \Delta_t$ if and only if, for every Δ_n , $1 \leq n \leq t$, every ω_i required by Δ_n is provided by some Δ_k with $k < n$.*

The “only if” part of the equivalence is immediate from the preceding discussion: the sequence μ_0, μ_1, \dots starts at $\mu_0 = 0$, and if it moves towards any wall of the Weyl chamber before first moving away from it, it will pass through the wall and some μ_i will be outside the chamber. Conversely, if $\Delta_1 \dots \Delta_t$ is any path-type which satisfies the condition of the lemma, then $\Delta_1^{m_1} \dots \Delta_t^{m_t}$ will definitely appear in the tree when $m_1 \gg m_2 \gg \dots \gg m_t$. This ensures that the coordinates of the μ_i are always nonnegative, since the sequence of μ s moves sufficiently far away from any wall of the Weyl chamber before the first time it moves back towards it. \square

We could compute the exact conditions on the m_i for a specific path; in general, they all require that m_n be bounded by some linear combination of m_1, \dots, m_{n-1} , and the first m_i appearing with nonzero coefficient in that linear combination has positive coefficient.

Now we can show that the number of nodes of path-type $\Delta_1 \dots \Delta_t$ appearing on the m th level of the tree grows as m^{t-1} . Consider the path $\Delta_1^{m_1} \dots \Delta_t^{m_t}$ as a point (m_1, \dots, m_t) in \mathbb{R}^t . The path ends on row m if $m = m_1 + \dots + m_t$, so solutions lie on a plane of dimension $t - 1$; the number of solutions to that equality in nonnegative integers is $\binom{m+t-1}{t-1}$, which certainly grows as m^{t-1} , as expected. The further linear inequalities on the m_i which ensure that μ_1, \dots, μ_m remain in the Weyl chamber correspond to hyperplanes through the origin which our solutions must lie on one side of, but the resulting region still has full dimension $t - 1$ since the generic point with $m_1 \gg m_2 \gg \dots \gg m_t$ satisfies all of the inequalities, as shown above.

The highest weight of the \mathfrak{g} -module at the node associated with the generic solution of the form $m_1 \gg m_2 \gg \dots \gg m_t$ grows linearly in m . Its dimension, therefore, grows as a polynomial in m , and the degree of the polynomial is just the number of positive roots of the Lie algebra which are not orthogonal to the highest weight. The only positive roots perpendicular to this generic highest weight are those perpendicular to every highest weight which comes from a path of type $\Delta_1 \dots \Delta_t$, and the number of such roots is the degree of polynomial growth of the dimensions of the representations of the \mathfrak{g} -module.

We can figure out how the multiplicities of nodes with a specific path-type grow as well. Theorem 3.2 gives a formula for multiplicities as a product of binomial

coefficients over $1 \leq k \leq r$ and $n \geq 1$. The only terms in the product which are not 1 correspond to nonzero values of $\delta_n - \delta_{n+1}$. In the path $\Delta_1^{m_1} \dots \Delta_t^{m_t}$, these occur only when $n = m_1 + \dots + m_i$ for some $1 \leq i \leq t$, so that $\delta_n - \delta_{n+1}$ is $\Delta_i - \Delta_{i+1}$ (where Δ_{t+1} is just 0). Following our previous notation, let $\delta_n - \delta_{n+1} = \mathbf{d}_n = \sum \mathbf{d}_n^{(k)} \alpha_k$. If we take any k for which $\mathbf{d}_n^{(k)}$ is nonzero, there are two possibilities for the contribution to the multiplicity from its binomial coefficient. If ω_k has been provided by at least one of $\Delta_1, \dots, \Delta_i$, then the value $P_n^{(k)}$ is a linear combination of m_1, \dots, m_i , which grows linearly as m gets large. In this case, the binomial coefficient grows as a polynomial in m of degree $\mathbf{d}_n^{(k)}$. On the other hand, if ω_k has not been provided, then the binomial coefficient is just 1.

For any k , $1 \leq k \leq r$, define $f(k)$ to be the smallest i in our path-type such that Δ_i provides ω_k ; we say that Δ_i provides ω_k for the first time. Then the total contribution to the multiplicity from the coordinate k will be the product of the contributions when $n = m_1 + \dots + m_j$ for $j = f(k), f(k) + 1, \dots, t$. As m gets large, the product of these contributions grows as a polynomial of degree $\sum_{j=f(k)}^t \mathbf{d}_n^{(m_1 + \dots + m_j)}$; that is, the sum of the decreases in the α_k -coordinate of the Δ s. But since Δ_{t+1} is just 0, that sum is exactly the α_k -coordinate of $\Delta_{f(k)}$.

So given a path-type $\Delta_1 \dots \Delta_t$ which Lemma 3.9 says appears in $T(\ell)$, the total of the multiplicities of the nodes of that path-type which appear in the top m rows of $T(\ell)$ grows as a polynomial of degree

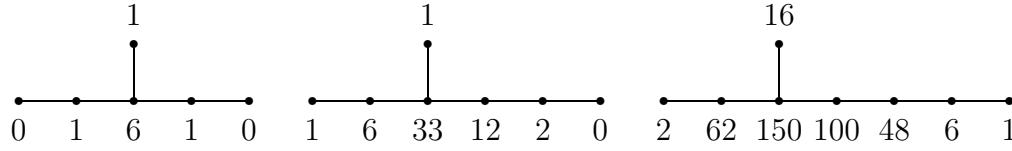
$$g(\Delta_1 \dots \Delta_t) = t + \sum_{k=1}^r \alpha_k\text{-coordinate of } \Delta_{f(k)} \quad (3.6)$$

where we take $\Delta_{f(k)}$ to be 0 if ω_k is not provided by any Δ in the path-type. This value is just the sum of the degrees of the polynomial growths described above.

Finally, since there are only finitely many path-types, the growth of the entire tree $T(\ell)$ is the same as the growth of the part corresponding to any path-type $\Delta_1 \dots \Delta_t$ which maximizes $g(\Delta_1 \dots \Delta_t)$. So we have proven the following, up to some calculation:

Theorem 3.10 *Let \mathfrak{g} be simply-laced with decompositions of $W_m(\ell)$ given by Theorem 3.2. Then the dimension of the representation $W_m(\ell)$ as m gets large is asymptotic to a polynomial in m of degree $\text{perp}(\Delta_1 \dots \Delta_t) + g(\Delta_1 \dots \Delta_t)$, where the path-type $\Delta_1 \dots \Delta_t$ is one which maximizes the value of g , and $\text{perp}(\Delta_1 \dots \Delta_t)$ is the number of positive roots orthogonal to all highest weights of nodes with path-type $\Delta_1 \dots \Delta_t$.*

1. If \mathfrak{g} is of type A_n then the maximum value of $g(\Delta_1 \dots \Delta_t)$ is 0, for all $1 \leq \ell \leq n$.
2. If \mathfrak{g} is of type D_n then the maximum value of $g(\Delta_1 \dots \Delta_t)$ is $\lfloor \ell/2 \rfloor$, for $1 \leq \ell \leq n-2$, and 0 for $\ell = n-1, n$.
3. If \mathfrak{g} is of type E_6, E_7 , or E_8 , the maximum value of $g(\Delta_1 \dots \Delta_t)$ is



We will complete the proof by exhibiting the path-types which give the indicated values of g and proving they are maximal.

If $\Delta_1 \dots \Delta_t$ maximizes the value of g , then it cannot be obtained from any other path-type by inserting an extra Δ , since any insertion would increase the length t and would not decrease the sum in the definition of g . Therefore each Δ_k in our desired path-type must be in the positive root lattice, allowable according to Lemma 3.9, and must be maximal (under \preceq) in meeting those requirements; we will call a path-type maximal if this is the case.

In particular, if ω_ℓ is in the root lattice then Δ_1 will be ω_ℓ , and a g -value of 0 corresponds exactly to an ω_ℓ which is not in the root lattice and is a minimal weight. Thus the 0s above can be verified by inspection; these are exactly the cases in which $W_m(\ell)$ remains irreducible as a \mathfrak{g} -module. Similarly, if ω_ℓ is not in the root lattice but there is only one point in the lattice and in the Weyl chamber under ω_ℓ , the path-type will consist just of that point. We can now limit ourselves to path-types of length greater than one.

If \mathfrak{g} is of type D_n then for each ω_ℓ , $2 \leq \ell \leq n-2$, there is a unique maximal path-type:

$$\begin{aligned} \omega_\ell \succ \omega_\ell - \omega_2 \succ \omega_\ell - \omega_4 \succ \dots \succ \omega_\ell - \omega_{\ell-2} & \quad \text{when } \ell \text{ is even} \\ \omega_\ell - \omega_1 \succ \omega_\ell - \omega_3 \succ \dots \succ \omega_\ell - \omega_{\ell-2} & \quad \text{when } \ell \text{ is odd} \end{aligned}$$

In both cases, the only contribution to g comes from the length of the path, which is $\lfloor \ell/2 \rfloor$. This also means that the nodes of the tree $T(\ell)$ will all have multiplicity 1 in this case. See figure 3.4 (p. 30) for a graphical interpretation of this path-type.

When \mathfrak{g} is of type E_6 , E_7 or E_8 , the following weights have a unique maximal path-type (of length > 1), whose g -value is given in Theorem 3.10:

$$\begin{aligned}
E_6 \quad \ell = 4 \quad & \omega_4 \succ \omega_4 - \omega_2 \succ \omega_4 - \omega_1 - \omega_6 \succ \omega_2 + \omega_4 - \omega_3 - \omega_5 \succ 2\omega_2 - \omega_4 \\
E_7 \quad \ell = 3 \quad & \omega_3 \succ \omega_3 - \omega_1 \succ \omega_3 - \omega_6 \succ \omega_1 + \omega_6 - \omega_4 \succ 2\omega_1 - \omega_3 \\
& \ell = 6 \quad \omega_6 \succ \omega_6 - \omega_1 \\
E_8 \quad \ell = 1 \quad & \omega_1 \succ \omega_1 - \omega_8 \\
& \ell = 7 \quad \omega_7 \succ \omega_7 - \omega_8 \succ \omega_7 - \omega_1 \succ \omega_7 + \omega_8 - \omega_6 \succ 2\omega_8 - \omega_7 \\
& \ell = 8 \quad \omega_8
\end{aligned}$$

We consider the remaining weights in E_8 next. Consider the incomplete path-type

$$\omega_\ell \succ \omega_\ell - \omega_8 \succ \omega_\ell - \omega_1 \succ \omega_\ell - \omega_6 + \omega_8 \succ \omega_\ell + \omega_1 - \omega_4 + \omega_8 \succ \dots$$

where ω_ℓ is any fundamental weight which is in the root lattice and high enough that all of the weights in question lie in the Weyl chamber. The path so far provides ω_8 , ω_1 , ω_6 and ω_4 ; notice that for any ω_i which has not been provided, all of its neighbors in the Dynkin diagram have. Therefore we can extend this path four more steps by subtracting one of α_2 , α_3 , α_5 and α_7 at each step, to produce a path in which every ω_i has been provided. This can be extended to a full path-type by subtracting any α_i at each stage until we reach the walls of the Weyl chamber.

The resulting path-type is maximal, and is the unique maximal one up to a sequence of transformations of the form

$$\dots \succ \Delta \succ \Delta - \lambda \succ \Delta - \lambda - \mu \succ \dots \mapsto \dots \succ \Delta \succ \Delta - \mu \succ \Delta - \lambda - \mu \succ \dots$$

which do not affect the rate of growth g . All relevant weights are in the Weyl chamber if and only if $\omega_\ell \succ \xi = (4, 8, 10, 14, 12, 8, 6, 2)$; this turns out to be everything except ω_1 , ω_7 and ω_8 , whose path-types are given above. If the path-type could start at ξ , it would have growth $g = 8$, though this is not possible since the last weight in the path-type would be 0 in this case. But each increase of the starting point of the path by any α_i increases g by 2 (1 from the length of the path and 1 from the multiplicity). So the growth for any $\omega_\ell \succ \xi$ is a linear function of its height with coefficient 2; $g = 2\text{ht}(\omega_\ell) - 120$.

The only remaining cases are ω_4 and ω_5 when \mathfrak{g} is of type E_7 . Both work like the general case for E_8 , beginning instead with the incomplete path-types

$$\begin{aligned}
\ell = 4 \quad & \omega_4 \succ \omega_4 - \omega_1 \succ \omega_4 - \omega_6 \succ \omega_1 \succ \dots \\
\ell = 5 \quad & \omega_5 - \omega_7 \succ \omega_5 - \omega_2 \succ \omega_5 + \omega_7 - \omega_1 - \omega_6 \succ \omega_2 + \omega_7 - \omega_3 \succ \dots
\end{aligned}$$

This concludes the proof of Theorem 3.10. \square

The same argument used for E_8 shows that for any choice of \mathfrak{g} , all “sufficiently large” weights ω_ℓ in a particular translate of the root lattice will have growth given by $2\text{ht}(\omega_\ell) - c$ for some fixed c . A weight is sufficiently large if every ω_i is provided in its maximal path. Thus we can easily check that ω_4 and ω_5 qualify for E_7 , and in both cases $c = 63$. Similarly, ω_4 for E_6 qualifies, and $c = 36$. While there are no sufficiently large fundamental weights for A_n or D_n , we can compute what the maximal path-type would be if one did exist, and in all cases, c is the number of positive roots. A uniform explanation would be nice, though the exhaustive computation does provide a complete proof.

3.5 Computations

This section gives the decompositions of $W_m(\ell)$ into \mathfrak{g} -modules predicted by the conjectural formulas in [KR2] for D_n and E_n . We also give the tree structure defined in Section 3.2. Roots and weights are numbered as in Table 2.1 (p. 12). As already noted, when \mathfrak{g} is of type A_n , the $Y(\mathfrak{g})$ -modules $W_m(\ell)$ remain irreducible when viewed as \mathfrak{g} -modules.

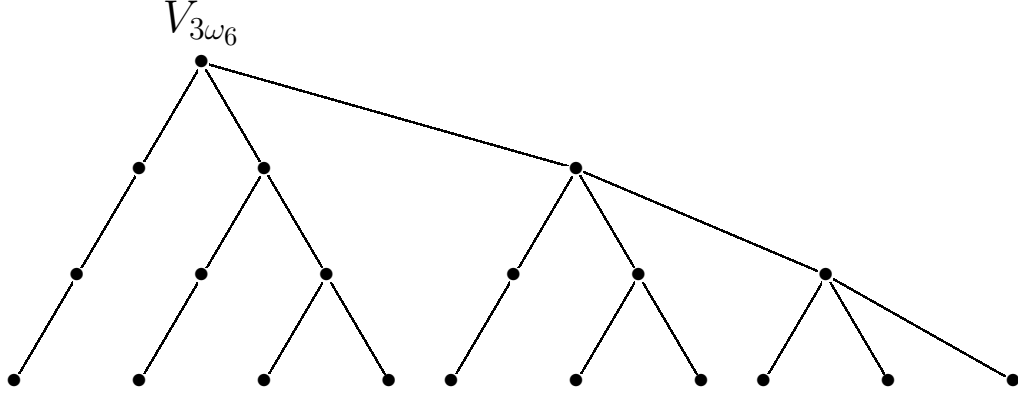
The representations $W_m(\ell)$ when $m = 1$ are called fundamental representations. In the setting of $U_q(\mathfrak{g})$ -module decompositions of $U_q(\hat{\mathfrak{g}})$ modules, the decompositions of the fundamental representations for all \mathfrak{g} and most choices of ω_ℓ appear in [ChP], calculated using techniques unrelated to the conjecture used in [KR2] to give formulas (3.2) and (3.3). Those computations agree with the ones given below.

The choices of ω_ℓ not calculated in [ChP] are exactly those in which the maximal path-type (Theorem 3.10) is not unique.

D_n

Let \mathfrak{g} be of type D_n . As already noted, the fundamental weights ω_{n-1} and ω_n are minimal with respect to \preceq , so $W_m(n-1)$ and $W_m(n)$ remain irreducible as \mathfrak{g} -modules. Now suppose $\ell \leq n-2$. Then the structure of the weights in the Weyl chamber under ω_ℓ does not depend on n , and so the decomposition of $W_m(\ell)$ in D_n is the same for any $n \geq \ell+2$.

As mentioned in the proof of Theorem 3.10, there is a unique maximal path-type for each ω_ℓ , and there are no multiplicities greater than 1. The decomposition is



The children of a node with highest weight λ have highest weights $\lambda - \omega_6 + \omega_4$, $\lambda - \omega_6 + \omega_2$, and $\lambda - \omega_6$, if they are to the left, right, or far right of the parent, respectively.

The tree for $W_3(7)$ looks identical, but with $V_{3\omega_7}$ on top and highest weights $\lambda - \omega_7 + \omega_5$, $\lambda - \omega_7 + \omega_3$, and $\lambda - \omega_5 + \omega_1$ for the children.

Figure 3.4: Tree structure of the decomposition of $W_3(6)$ for D_n for any $n \geq 8$.

therefore very simple: if $\ell \leq n - 2$ is even, then

$$W_m(\ell) \simeq \bigoplus_{k_2+k_4+\dots+k_{\ell-2}+k_\ell=k \leq m} V_{k_2\omega_2+k_4\omega_4+\dots+k_{\ell-2}\omega_{\ell-2}+(m-k)\omega_\ell}$$

and if ℓ is odd, then

$$W_m(\ell) \simeq \bigoplus_{k_1+k_3+\dots+k_{\ell-2}=k \leq m} V_{k_1\omega_1+k_3\omega_3+\dots+k_{\ell-2}\omega_{\ell-2}+(m-k)\omega_\ell}$$

The minor difference is because ω_ℓ for ℓ odd is not in the root lattice. The sum k is the level of the tree on which that module appears, and the parent of a module is obtained by subtracting 1 from the first of $k_{\ell-2}, k_{\ell-4}, \dots$ which is nonzero (or from k_ℓ if nothing else is nonzero and ℓ is even).

Figure 3.4 illustrates the tree structure for $W_m(\ell)$ when $\ell = 6$ or $\ell = 7$ (and $n \geq \ell + 2$). For $\ell = 4$ or 5 , the tree would look like the left-most triangle of figure 3.4, while for 8 or 9 the evident recursive pattern would be one level deeper.

E_n

When \mathfrak{g} is of type E_n the tree structure is much more irregular: these are the only cases in which a \mathfrak{g} -module can appear in more than one place in the tree and in

which a node on the tree can have multiplicity greater than one.

We indicate the tree structure as follows: we list every node in the tree, starting with the root and in depth-first order, and a node on level k of the tree is written as $\overset{k}{\oplus} V_\lambda$. This is enough information to recover the entire tree, since the parent of that node is the most recent summand of the form $\overset{k-1}{\oplus} V_\mu$. Comparing Figure 3.1 to its representation here should make the notation clear.

Due to space considerations, for E_6 we list calculations for $m \leq 3$, for E_7 we list $m \leq 2$, and for E_8 only $m = 1$. The tree decomposition for $W_3(4)$ for E_7 , for example, would have 836 components.

E_6

$W_m(1)$ remains irreducible for all m .

$$W_1(2) \simeq V_{\omega_2} \overset{1}{\oplus} V_0$$

$$W_2(2) \simeq V_{2\omega_2} \overset{1}{\oplus} V_{\omega_2} \overset{2}{\oplus} V_0$$

$$W_3(2) \simeq V_{3\omega_2} \overset{1}{\oplus} V_{2\omega_2} \overset{2}{\oplus} V_{\omega_2} \overset{3}{\oplus} V_0$$

$$W_1(3) \simeq V_{\omega_3} \overset{1}{\oplus} V_{\omega_6}$$

$$W_2(3) \simeq V_{2\omega_3} \overset{1}{\oplus} V_{\omega_3+\omega_6} \overset{2}{\oplus} V_{2\omega_6}$$

$$W_3(3) \simeq V_{3\omega_3} \overset{1}{\oplus} V_{2\omega_3+\omega_6} \overset{2}{\oplus} V_{\omega_3+2\omega_6} \overset{3}{\oplus} V_{3\omega_6}$$

$$W_1(4) \simeq V_{\omega_4} \overset{1}{\oplus} V_{\omega_1+\omega_6} \overset{1}{\oplus} 2V_{\omega_2} \overset{1}{\oplus} V_0$$

$$W_2(4) \simeq V_{2\omega_4} \overset{1}{\oplus} V_{\omega_1+\omega_4+\omega_6} \overset{2}{\oplus} V_{2\omega_1+2\omega_6} \overset{1}{\oplus} 2V_{\omega_2+\omega_4} \overset{2}{\oplus} V_{\omega_3+\omega_5} \overset{2}{\oplus} 2V_{\omega_1+\omega_2+\omega_6} \overset{2}{\oplus} 3V_{2\omega_2} \overset{3}{\oplus} V_{\omega_4} \overset{1}{\oplus} V_{\omega_4} \overset{2}{\oplus} V_{\omega_1+\omega_6} \overset{2}{\oplus} 2V_{\omega_2} \overset{2}{\oplus} V_0$$

$$W_3(4) \simeq V_{3\omega_4} \overset{1}{\oplus} V_{\omega_1+2\omega_4+\omega_6} \overset{2}{\oplus} V_{2\omega_1+\omega_4+2\omega_6} \overset{3}{\oplus} V_{3\omega_1+3\omega_6} \overset{1}{\oplus} 2V_{\omega_2+2\omega_4} \overset{2}{\oplus} V_{\omega_3+\omega_4+\omega_5} \overset{2}{\oplus} 2V_{\omega_1+\omega_2+\omega_4+\omega_6} \overset{3}{\oplus} V_{\omega_1+\omega_3+\omega_5+\omega_6} \overset{3}{\oplus} 2V_{2\omega_1+\omega_2+2\omega_6} \overset{2}{\oplus} 3V_{2\omega_2+\omega_4} \overset{3}{\oplus} V_{2\omega_4} \overset{3}{\oplus} 2V_{\omega_2+\omega_3+\omega_5} \overset{3}{\oplus} 3V_{\omega_1+2\omega_2+\omega_6} \overset{4}{\oplus} V_{\omega_1+\omega_4+\omega_6} \overset{3}{\oplus} 4V_{3\omega_2} \overset{4}{\oplus} 2V_{\omega_2+\omega_4} \overset{1}{\oplus} V_{2\omega_4} \overset{2}{\oplus} V_{\omega_1+\omega_4+\omega_6} \overset{3}{\oplus} V_{2\omega_1+2\omega_6} \overset{2}{\oplus} 2V_{\omega_2+\omega_4} \overset{3}{\oplus} V_{\omega_3+\omega_5} \overset{3}{\oplus} 2V_{\omega_1+\omega_2+\omega_6} \overset{3}{\oplus} 3V_{2\omega_2} \overset{4}{\oplus} V_{\omega_4} \overset{2}{\oplus} V_{\omega_4} \overset{3}{\oplus} V_{\omega_1+\omega_6} \overset{3}{\oplus} 2V_{\omega_2} \overset{3}{\oplus} V_0$$

$$W_1(5) \simeq V_{\omega_5} \overset{1}{\oplus} V_{\omega_1}$$

$$W_2(5) \simeq V_{2\omega_5} \overset{1}{\oplus} V_{\omega_1+\omega_5} \overset{2}{\oplus} V_{2\omega_1}$$

$$W_3(5) \simeq V_{3\omega_5} \overset{1}{\oplus} V_{\omega_1+2\omega_5} \overset{2}{\oplus} V_{2\omega_1+\omega_5} \overset{3}{\oplus} V_{3\omega_1}$$

$W_m(6)$ remains irreducible for all m .

E_7

$$W_1(1) \simeq V_{\omega_1} \overset{1}{\oplus} V_0$$

$$W_2(1) \simeq V_{2\omega_1} \overset{1}{\oplus} V_{\omega_1} \overset{2}{\oplus} V_0$$

$$W_1(2) \simeq V_{\omega_2} \overset{1}{\oplus} V_{\omega_7}$$

$$W_2(2) \simeq V_{2\omega_2} \overset{1}{\oplus} V_{\omega_2+\omega_7} \overset{2}{\oplus} V_{2\omega_7}$$

$$W_1(3) \simeq V_{\omega_3} \overset{1}{\oplus} V_{\omega_6} \overset{1}{\oplus} 2V_{\omega_1} \overset{1}{\oplus} V_0$$

$$W_2(3) \simeq V_{2\omega_3} \overset{1}{\oplus} V_{\omega_3+\omega_6} \overset{2}{\oplus} V_{2\omega_6} \overset{1}{\oplus} 2V_{\omega_1+\omega_3} \overset{2}{\oplus} V_{\omega_4} \overset{2}{\oplus} 2V_{\omega_1+\omega_6} \overset{2}{\oplus} 3V_{2\omega_1} \overset{3}{\oplus} V_{\omega_3} \overset{1}{\oplus} V_{\omega_3} \overset{2}{\oplus} V_{\omega_6} \overset{2}{\oplus} 2V_{\omega_1} \overset{2}{\oplus} V_0$$

$$W_1(4) \simeq V_{\omega_4} \overset{1}{\oplus} V_{\omega_1+\omega_6} \overset{1}{\oplus} 2V_{\omega_2+\omega_7} \overset{1}{\oplus} V_{2\omega_1} \overset{1}{\oplus} 3V_{\omega_3} \overset{2}{\oplus} V_{\omega_6} \overset{1}{\oplus} V_{2\omega_7} \overset{1}{\oplus} 3V_{\omega_6} \overset{2}{\oplus} V_{\omega_1} \overset{1}{\oplus} 3V_{\omega_1} \overset{2}{\oplus} V_0 \overset{1}{\oplus} V_0$$

$$\begin{aligned} W_2(4) \simeq & V_{2\omega_4} \overset{1}{\oplus} V_{\omega_1+\omega_4+\omega_6} \overset{2}{\oplus} V_{2\omega_1+2\omega_6} \overset{1}{\oplus} 2V_{\omega_2+\omega_4+\omega_7} \overset{2}{\oplus} V_{\omega_3+\omega_5+\omega_7} \overset{2}{\oplus} 2V_{\omega_1+\omega_2+\omega_6+\omega_7} \overset{2}{\oplus} \\ & 3V_{2\omega_2+2\omega_7} \overset{3}{\oplus} V_{\omega_4+2\omega_7} \overset{1}{\oplus} V_{2\omega_1+\omega_4} \overset{2}{\oplus} V_{3\omega_1+\omega_6} \overset{2}{\oplus} V_{4\omega_1} \overset{1}{\oplus} 3V_{\omega_3+\omega_4} \overset{2}{\oplus} 2V_{\omega_1+\omega_2+\omega_5} \overset{2}{\oplus} \\ & 4V_{\omega_1+\omega_3+\omega_6} \overset{2}{\oplus} V_{2\omega_5} \overset{2}{\oplus} V_{2\omega_2+\omega_6} \overset{2}{\oplus} 3V_{\omega_4+\omega_6} \overset{3}{\oplus} V_{\omega_1+2\omega_6} \overset{2}{\oplus} 2V_{2\omega_1+\omega_2+\omega_7} \overset{2}{\oplus} 6V_{\omega_2+\omega_3+\omega_7} \overset{3}{\oplus} \\ & 2V_{\omega_1+\omega_5+\omega_7} \overset{3}{\oplus} 2V_{\omega_2+\omega_6+\omega_7} \overset{2}{\oplus} 3V_{2\omega_1+\omega_3} \overset{2}{\oplus} 6V_{2\omega_3} \overset{3}{\oplus} 3V_{\omega_1+\omega_4} \overset{3}{\oplus} V_{\omega_2+\omega_5} \overset{3}{\oplus} V_{2\omega_1+\omega_6} \overset{3}{\oplus} \\ & 3V_{\omega_3+\omega_6} \overset{4}{\oplus} V_{2\omega_6} \overset{1}{\oplus} V_{\omega_4+2\omega_7} \overset{2}{\oplus} V_{\omega_1+\omega_6+2\omega_7} \overset{2}{\oplus} 2V_{\omega_2+3\omega_7} \overset{2}{\oplus} V_{4\omega_7} \overset{1}{\oplus} 3V_{\omega_4+\omega_6} \overset{2}{\oplus} 2V_{\omega_2+\omega_3+\omega_7} \overset{2}{\oplus} \\ & 3V_{\omega_1+2\omega_6} \overset{2}{\oplus} 4V_{\omega_1+\omega_5+\omega_7} \overset{2}{\oplus} V_{2\omega_3} \overset{2}{\oplus} V_{\omega_1+2\omega_2} \overset{2}{\oplus} 3V_{\omega_1+\omega_4} \overset{3}{\oplus} V_{2\omega_1+\omega_6} \overset{2}{\oplus} 8V_{\omega_2+\omega_6+\omega_7} \overset{3}{\oplus} \\ & 2V_{\omega_3+2\omega_7} \overset{2}{\oplus} 6V_{\omega_2+\omega_5} \overset{3}{\oplus} 2V_{\omega_3+\omega_6} \overset{3}{\oplus} 2V_{\omega_1+\omega_2+\omega_7} \overset{2}{\oplus} V_{2\omega_1+2\omega_7} \overset{2}{\oplus} 3V_{\omega_3+2\omega_7} \overset{3}{\oplus} V_{\omega_6+2\omega_7} \overset{2}{\oplus} \\ & 3V_{2\omega_1+\omega_6} \overset{3}{\oplus} V_{3\omega_1} \overset{2}{\oplus} 9V_{\omega_3+\omega_6} \overset{3}{\oplus} 4V_{\omega_1+\omega_2+\omega_7} \overset{3}{\oplus} 3V_{2\omega_6} \overset{3}{\oplus} 4V_{\omega_5+\omega_7} \overset{3}{\oplus} 4V_{\omega_1+\omega_3} \overset{3}{\oplus} V_{2\omega_2} \overset{3}{\oplus} \\ & 3V_{\omega_4} \overset{4}{\oplus} V_{\omega_1+\omega_6} \overset{2}{\oplus} 3V_{\omega_6+2\omega_7} \overset{2}{\oplus} 6V_{2\omega_6} \overset{3}{\oplus} 3V_{\omega_5+\omega_7} \overset{3}{\oplus} V_{\omega_4} \overset{3}{\oplus} V_{\omega_1+2\omega_7} \overset{3}{\oplus} 3V_{\omega_1+\omega_6} \overset{4}{\oplus} V_{2\omega_1} \overset{1}{\oplus} \\ & 3V_{\omega_1+\omega_4} \overset{2}{\oplus} 2V_{\omega_2+\omega_5} \overset{2}{\oplus} 3V_{2\omega_1+\omega_6} \overset{2}{\oplus} 4V_{\omega_3+\omega_6} \overset{2}{\oplus} 8V_{\omega_1+\omega_2+\omega_7} \overset{3}{\oplus} 2V_{\omega_5+\omega_7} \overset{2}{\oplus} V_{2\omega_6} \overset{2}{\oplus} 3V_{\omega_5+\omega_7} \overset{3}{\oplus} \end{aligned}$$

$$\begin{aligned}
& V_{\omega_1+2\omega_7} \oplus^2 3V_{3\omega_1} \oplus^2 12V_{\omega_1+\omega_3} \oplus^3 4V_{\omega_4} \oplus^3 4V_{\omega_1+\omega_6} \oplus^2 3V_{2\omega_2} \oplus^3 V_{\omega_4} \oplus^2 5V_{\omega_4} \oplus^3 3V_{\omega_1+\omega_6} \oplus^3 \\
& 4V_{\omega_2+\omega_7} \oplus^3 V_{2\omega_1} \oplus^3 3V_{\omega_3} \oplus^4 V_{\omega_6} \oplus^2 3V_{\omega_1+2\omega_7} \oplus^2 9V_{\omega_1+\omega_6} \oplus^3 4V_{\omega_2+\omega_7} \oplus^3 3V_{2\omega_1} \oplus^3 4V_{\omega_3} \oplus^3 \\
& V_{2\omega_7} \oplus^3 3V_{\omega_6} \oplus^4 V_{\omega_1} \oplus^2 6V_{2\omega_1} \oplus^3 3V_{\omega_3} \oplus^3 V_{\omega_6} \oplus^3 3V_{\omega_1} \oplus^4 V_0 \oplus^1 V_{\omega_4} \oplus^2 V_{\omega_1+\omega_6} \oplus^2 2V_{\omega_2+\omega_7} \oplus^2 \\
& V_{2\omega_1} \oplus^2 3V_{\omega_3} \oplus^3 V_{\omega_6} \oplus^2 V_{2\omega_7} \oplus^2 3V_{\omega_6} \oplus^3 V_{\omega_1} \oplus^2 3V_{\omega_1} \oplus^3 V_0 \oplus^2 V_0
\end{aligned}$$

$$W_1(5) \simeq V_{\omega_5} \oplus^1 V_{\omega_1+\omega_7} \oplus^1 2V_{\omega_2} \oplus^1 2V_{\omega_7}$$

$$\begin{aligned}
W_2(5) \simeq & V_{2\omega_5} \oplus^1 V_{\omega_1+\omega_5+\omega_7} \oplus^2 V_{2\omega_1+2\omega_7} \oplus^1 2V_{\omega_2+\omega_5} \oplus^2 V_{\omega_3+\omega_6} \oplus^2 2V_{\omega_1+\omega_2+\omega_7} \oplus^2 3V_{2\omega_2} \oplus^3 V_{\omega_4} \oplus^1 \\
& 2V_{\omega_5+\omega_7} \oplus^2 V_{\omega_4} \oplus^2 2V_{\omega_1+2\omega_7} \oplus^2 2V_{\omega_1+\omega_6} \oplus^2 4V_{\omega_2+\omega_7} \oplus^3 V_{\omega_3} \oplus^2 3V_{2\omega_7} \oplus^3 V_{\omega_6}
\end{aligned}$$

$$W_1(6) \simeq V_{\omega_6} \oplus^1 V_{\omega_1} \oplus^1 V_0$$

$$W_2(6) \simeq V_{2\omega_6} \oplus^1 V_{\omega_1+\omega_6} \oplus^2 V_{2\omega_1} \oplus^1 V_{\omega_6} \oplus^2 V_{\omega_1} \oplus^2 V_0$$

$W_m(7)$ remains irreducible for all m .

E_8

$$W_1(1) \simeq V_{\omega_1} \oplus^1 V_{\omega_8} \oplus^1 V_0$$

$$W_1(2) \simeq V_{\omega_2} \oplus^1 V_{\omega_7} \oplus^1 2V_{\omega_1} \oplus^1 2V_{\omega_8} \oplus^1 V_0$$

$$W_1(3) \simeq V_{\omega_3} \oplus^1 V_{\omega_6} \oplus^1 2V_{\omega_1+\omega_8} \oplus^1 3V_{\omega_2} \oplus^2 V_{\omega_7} \oplus^1 V_{2\omega_8} \oplus^1 3V_{\omega_7} \oplus^2 V_{\omega_1} \oplus^1 4V_{\omega_1} \oplus^2 2V_{\omega_8} \oplus^1 3V_{\omega_8} \oplus^2 V_0 \oplus^1 V_0$$

$$\begin{aligned}
W_1(4) \simeq & V_{\omega_4} \oplus^1 V_{\omega_1+\omega_6} \oplus^1 2V_{\omega_2+\omega_7} \oplus^1 V_{2\omega_1+\omega_8} \oplus^1 3V_{\omega_3+\omega_8} \oplus^2 V_{\omega_6+\omega_8} \oplus^1 V_{2\omega_7} \oplus^1 6V_{\omega_1+\omega_2} \oplus^2 \\
& 2V_{\omega_5} \oplus^2 2V_{\omega_1+\omega_7} \oplus^1 3V_{\omega_6+\omega_8} \oplus^2 V_{\omega_1+2\omega_8} \oplus^1 5V_{\omega_5} \oplus^2 3V_{\omega_1+\omega_7} \oplus^2 4V_{\omega_2+\omega_8} \oplus^2 V_{2\omega_1} \oplus^2 3V_{\omega_3} \oplus^3 \\
& V_{\omega_6} \oplus^1 3V_{\omega_1+2\omega_8} \oplus^2 V_{3\omega_8} \oplus^1 9V_{\omega_1+\omega_7} \oplus^2 4V_{\omega_2+\omega_8} \oplus^2 3V_{2\omega_1} \oplus^2 4V_{\omega_3} \oplus^2 4V_{\omega_7+\omega_8} \oplus^2 3V_{\omega_6} \oplus^3 \\
& V_{\omega_1+\omega_8} \oplus^1 10V_{\omega_2+\omega_8} \oplus^2 4V_{\omega_3} \oplus^2 6V_{\omega_7+\omega_8} \oplus^2 6V_{\omega_6} \oplus^2 8V_{\omega_1+\omega_8} \oplus^3 2V_{\omega_2} \oplus^1 6V_{2\omega_1} \oplus^2 3V_{\omega_3} \oplus^2 \\
& V_{\omega_6} \oplus^2 3V_{\omega_1+\omega_8} \oplus^3 V_{2\omega_8} \oplus^1 7V_{\omega_3} \oplus^2 5V_{\omega_6} \oplus^2 8V_{\omega_1+\omega_8} \oplus^2 9V_{\omega_2} \oplus^3 3V_{\omega_7} \oplus^2 V_{2\omega_8} \oplus^2 3V_{\omega_7} \oplus^3 \\
& V_{\omega_1} \oplus^1 V_{3\omega_8} \oplus^1 8V_{\omega_7+\omega_8} \oplus^2 3V_{\omega_6} \oplus^2 4V_{\omega_1+\omega_8} \oplus^2 3V_{2\omega_8} \oplus^3 V_{\omega_7} \oplus^1 7V_{\omega_6} \oplus^2 5V_{\omega_1+\omega_8} \oplus^2 8V_{\omega_2} \oplus^2 \\
& 3V_{2\omega_8} \oplus^2 9V_{\omega_7} \oplus^3 3V_{\omega_1} \oplus^3 3V_{\omega_1} \oplus^1 V_{\omega_8} \oplus^1 15V_{\omega_1+\omega_8} \oplus^2 8V_{\omega_2} \oplus^2 9V_{2\omega_8} \oplus^2 12V_{\omega_7} \oplus^2 9V_{\omega_1} \oplus^3 \\
& 3V_{\omega_8} \oplus^2 3V_{\omega_8} \oplus^3 V_0 \oplus^1 8V_{\omega_2} \oplus^2 6V_{\omega_7} \oplus^2 10V_{\omega_1} \oplus^2 6V_{\omega_8} \oplus^1 6V_{2\omega_8} \oplus^2 3V_{\omega_7} \oplus^2 V_{\omega_1} \oplus^2 3V_{\omega_8} \oplus^3 \\
& V_0 \oplus^1 7V_{\omega_7} \oplus^2 5V_{\omega_1} \oplus^2 8V_{\omega_8} \oplus^2 V_0 \oplus^1 7V_{\omega_1} \oplus^2 5V_{\omega_8} \oplus^2 3V_0 \oplus^1 5V_{\omega_8} \oplus^2 3V_0 \oplus^1 V_0
\end{aligned}$$

$$\begin{aligned}
W_1(5) \simeq & V_{\omega_5} \overset{1}{\oplus} V_{\omega_1+\omega_7} \overset{1}{\oplus} 2V_{\omega_2+\omega_8} \overset{1}{\oplus} V_{2\omega_1} \overset{1}{\oplus} 3V_{\omega_3} \overset{2}{\oplus} V_{\omega_6} \overset{1}{\oplus} 2V_{\omega_7+\omega_8} \overset{1}{\oplus} 4V_{\omega_6} \overset{2}{\oplus} 2V_{\omega_1+\omega_8} \overset{2}{\oplus} \\
& 2V_{\omega_2} \overset{1}{\oplus} 6V_{\omega_1+\omega_8} \overset{2}{\oplus} 2V_{\omega_2} \overset{2}{\oplus} 2V_{2\omega_8} \overset{2}{\oplus} 2V_{\omega_7} \overset{1}{\oplus} 5V_{\omega_2} \overset{2}{\oplus} 3V_{\omega_7} \overset{2}{\oplus} 4V_{\omega_1} \overset{1}{\oplus} 3V_{2\omega_8} \overset{2}{\oplus} V_{\omega_7} \overset{1}{\oplus} \\
& 5V_{\omega_7} \overset{2}{\oplus} 3V_{\omega_1} \overset{2}{\oplus} 4V_{\omega_8} \overset{1}{\oplus} 5V_{\omega_1} \overset{2}{\oplus} 3V_{\omega_8} \overset{2}{\oplus} V_0 \overset{1}{\oplus} 4V_{\omega_8} \overset{2}{\oplus} 2V_0 \overset{1}{\oplus} V_0
\end{aligned}$$

$$W_1(6) \simeq V_{\omega_6} \overset{1}{\oplus} V_{\omega_1+\omega_8} \overset{1}{\oplus} 2V_{\omega_2} \overset{1}{\oplus} V_{2\omega_8} \overset{1}{\oplus} 3V_{\omega_7} \overset{2}{\oplus} V_{\omega_1} \overset{1}{\oplus} 3V_{\omega_1} \overset{2}{\oplus} V_{\omega_8} \overset{1}{\oplus} 3V_{\omega_8} \overset{2}{\oplus} V_0 \overset{1}{\oplus} V_0$$

$$W_1(7) \simeq V_{\omega_7} \overset{1}{\oplus} V_{\omega_1} \overset{1}{\oplus} 2V_{\omega_8} \overset{1}{\oplus} V_0$$

$$W_1(8) \simeq V_{\omega_8} \overset{1}{\oplus} V_0$$

Chapter 4

Polynomial Relations Among Characters

In this chapter we investigate some polynomial relations which appear to hold among the characters of certain finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$. Conjecture 2.6, which was the basis for Chapter 3, proposes a formula for these characters which does indeed satisfy these relations. (While this chapter is phrased in the language of $U_q(\hat{\mathfrak{g}})$, the statements in [KR2] are all in the language of Yangians; see the end of section 2.1 for the sad story of this translation.)

The main result of this chapter is that these polynomial relations have only one solution, using a positivity condition on characters of $U_q(\hat{\mathfrak{g}})$. Therefore a proof that the characters of $U_q(\hat{\mathfrak{g}})$ do indeed satisfy these relations would imply Conjecture 2.6.

4.1 Introduction

Retain the notions of the previous chapter: \mathfrak{g} is a complex finite-dimensional simple Lie algebra, $\hat{\mathfrak{g}}$ its corresponding affine Lie algebra. Because of the inclusion of quantum enveloping algebras $U_q(\mathfrak{g}) \hookrightarrow U_q(\hat{\mathfrak{g}})$, any finite-dimensional representation of $U_q(\hat{\mathfrak{g}})$ is a direct sum of irreducible representations of $U_q(\mathfrak{g})$.

Again we let $W_m(\ell)$ denote the representation whose Drinfeld polynomials were given in Definition 3.1, though this time it is a representation of $U_q(\hat{\mathfrak{g}})$. Let $Q_m(\ell)$ denote its character. These characters appear to satisfy certain remarkable polynomial identities. When \mathfrak{g} is simply-laced, the identities have the simple form

$$Q_m(\ell)^2 = Q_{m-1}(\ell) Q_{m+1}(\ell) + \prod_{\ell' \sim \ell} Q_m(\ell') \quad (4.1)$$

for each $\ell = 1, \dots, n$ and $m \geq 1$. The product is taken over all ℓ' adjacent to ℓ in the Dynkin diagram of \mathfrak{g} . Using these relations, it is possible to write any character $Q_m(\ell)$ in terms of the characters $Q_1(\ell)$ of the “fundamental representations” of $U_q(\hat{\mathfrak{g}})$.

The main result of this chapter is that these equations have only one solution where $Q_m(\ell)$ is the character of a $U_q(\mathfrak{g})$ -module with highest weight $m\omega_\ell$; that is, where $Q_m(\ell)$ is a positive integer linear combination of irreducible $U_q(\mathfrak{g})$ -characters with highest weights sitting under $m\omega_\ell$. (We restrict our attention to the classical families A_n , B_n , C_n and D_n .) We use the polynomial relations to write some of these multiplicities in terms of the multiplicities in the characters $Q_1(\ell)$, and the resulting inequalities determine all of the multiplicities.

4.2 Polynomial relations

We will study the characters $Q_m(\ell)$ of the finite-dimensional representations $W_m(\ell)$ of $U_q(\hat{\mathfrak{g}})$, where $m = 0, 1, 2, \dots$ and $\ell = 1, \dots, n$, where $n = \text{rank}(\mathfrak{g})$. Since $U_q(\mathfrak{g})$ appears as a Hopf subalgebra of $U_q(\hat{\mathfrak{g}})$, we can talk about weights and characters of $U_q(\hat{\mathfrak{g}})$ modules by restricting our attention to the $U_q(\mathfrak{g})$ action. From this point of view, $W_m(\ell)$ has highest weight $m\omega_\ell$. If $m = 0$ then $W_m(\ell)$ is the trivial representation and $Q_m(\ell) = 1$. The objects $W_1(\ell)$ and $Q_1(\ell)$ are called the *fundamental representations* and characters.

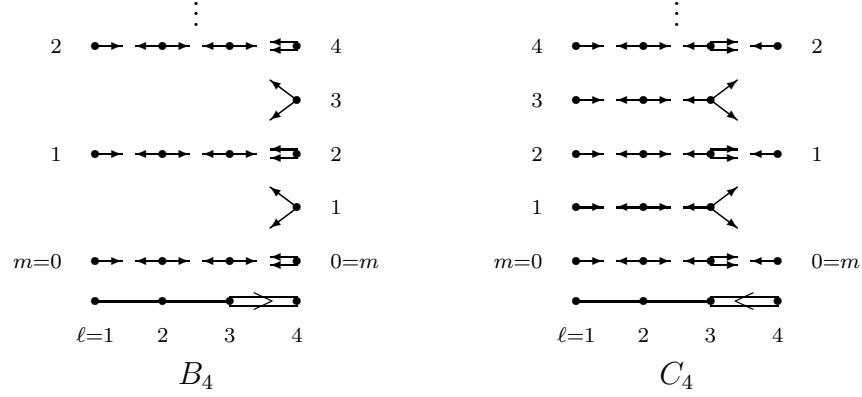
Let $V(\lambda)$ denote the character of the irreducible representation of $U_q(\mathfrak{g})$ with highest weight λ . We will write characters $Q_m(\ell)$ as sums $\sum m_\lambda V(\lambda)$. We will refer to the coefficients m_λ as the multiplicity of $V(\lambda)$ in the sum.

The characters $Q_m(\ell)$ when \mathfrak{g} is of type A_n are known to satisfy the so-called “discrete Hirota relations.” A conjectured generalization of these relations appears in [KR2] for the classical Lie algebras, and appear as the “ Q -system” in [KNS] for the exceptional cases as well. While we are only interested in the classical cases, we will give the relations in full generality.

For every positive integer m and for $\ell = 1, \dots, n$,

$$Q_m(\ell)^2 = Q_{m+1}(\ell) Q_{m-1}(\ell) + \prod_{\ell' \sim \ell} Q(m, \ell, \ell') \quad (4.2)$$

The product is over all ℓ' adjacent to ℓ in the Dynkin diagram of \mathfrak{g} , and the contribution $Q(m, \ell, \ell')$ from ℓ' is determined by the relative lengths of the roots α_ℓ and

Figure 4.1: Product from the definition of \mathcal{Q} for B_4 and C_4

$\alpha_{\ell'}$, as follows:

$$\mathcal{Q}(m, \ell, \ell') = \begin{cases} Q_m(\ell') & \text{if } \langle \alpha_\ell, \alpha_\ell \rangle = \langle \alpha_{\ell'}, \alpha_{\ell'} \rangle \\ Q_{km}(\ell') & \text{if } \langle \alpha_\ell, \alpha_\ell \rangle = k \langle \alpha_{\ell'}, \alpha_{\ell'} \rangle \\ \prod_{i=0}^{k-1} Q_{\lfloor \frac{m+i}{k} \rfloor}(\ell') & \text{if } k \langle \alpha_\ell, \alpha_\ell \rangle = \langle \alpha_{\ell'}, \alpha_{\ell'} \rangle \end{cases} \quad (4.3)$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x . We note that in the classical cases, the product differs from the simplified version in equation (4.1) only when:

$$\begin{aligned} \mathfrak{g} = \mathfrak{so}(2n+1), \quad \ell = n-1 : & \quad Q_m(n-2) Q_{2m}(n) \\ & \quad \ell = n : \quad Q_{\lfloor \frac{m}{2} \rfloor}(n-1) Q_{\lfloor \frac{m+1}{2} \rfloor}(n-1) \\ \mathfrak{g} = \mathfrak{sp}(2n), \quad \ell = n-1 : & \quad Q_m(n-2) Q_{\lfloor \frac{m}{2} \rfloor}(n) Q_{\lfloor \frac{m+1}{2} \rfloor}(n) \\ & \quad \ell = n : \quad Q_{2m}(n-1) \end{aligned}$$

The structure of the product is easily represented graphically, with a vertex for each character $Q_m(\ell)$ and an arrow from $Q_m(\ell)$ pointing at each term of $\prod \mathcal{Q}(m, \ell, \ell')$; see Figure 4.1 for \mathfrak{g} of type B_4 and C_4 . The corresponding picture for G_2 is similarly pleasing.

Finally, we can solve equation (4.2) to get a recurrence relation:

$$Q_m(\ell) = \frac{Q_{m-1}(\ell)^2 - \prod \mathcal{Q}(m-1, \ell, \ell')}{Q_{m-2}(\ell)} \quad (4.4)$$

Note that the recurrence is well-founded: repeated use eventually writes everything in terms of the fundamental characters $Q_1(\ell)$. This is just the statement that

iteration of “move down, then follow any arrow” in Figure 4.1 will eventually lead you from any point to one on the bottom row. In fact, $Q_m(\ell)$ is always a polynomial in the fundamental characters, though from looking at the recurrence it is only clear that it is a rational function. A Jacobi-Trudi style formula for writing the polynomial directly was given in [KNH].

The reason that characters of representations of quantum affine algebras are solutions to a discrete integrable system is still a bit of a mystery.

4.3 Main Theorem

The result of [KR2] was to conjecture a combinatorial formula for all the multiplicities $Z(m, \ell, \lambda)$ in the decomposition $Q_m(\ell) = \sum Z(m, \ell, \lambda)V(\lambda)$, reproduced as Conjecture 2.6 here. We will refer to these proposed characters as “combinatorial characters” of the representations $W_m(\ell)$, although it is unproven that they are the characters of some $U_q(\hat{\mathfrak{g}})$ module.

Theorem 4.1 (Kirillov-Reshetikhin) *Let \mathfrak{g} be of type A , B , C or D . The combinatorial characters of $W_m(\ell)$ are the unique solution to equations (4.2) and (4.3) with the initial data*

$$\begin{array}{ll} A_n : & Q_1(\ell) = V(\omega_\ell) \quad 1 \leq \ell \leq n \\ B_n : & Q_1(\ell) = V(\omega_\ell) + V(\omega_{\ell-2}) + V(\omega_{\ell-4}) + \cdots \quad 1 \leq \ell \leq n-1 \\ & Q_1(n) = V(\omega_n) \\ C_n : & Q_1(\ell) = V(\omega_\ell) \quad 1 \leq \ell \leq n \\ D_n : & Q_1(\ell) = V(\omega_\ell) + V(\omega_{\ell-2}) + V(\omega_{\ell-4}) + \cdots \quad 1 \leq \ell \leq n-2 \\ & Q_1(\ell) = V(\omega_\ell) \quad \ell = n-1, n \end{array}$$

The main result of this chapter is that the specification of initial data is unnecessary.

Theorem 4.2 *Let \mathfrak{g} be of type A , B , C or D . The combinatorial characters of the representations $W_m(\ell)$ are the only solutions to equations (4.2) and (4.3) such that $Q_m(\ell)$ is a character of a representation of $U_q(\mathfrak{g})$ with highest weight $m\omega_\ell$, for every nonnegative integer m and $1 \leq \ell \leq n$.*

We need only prove that any choice of initial data other than that in Theorem 4.1 would result in some $Q_m(\ell)$ which is not a character of a representation of $U_q(\mathfrak{g})$. The values $Q_m(\ell)$ are always virtual $U_q(\mathfrak{g})$ -characters, but in all other cases, some contain representations occurring with negative multiplicity. As an immediate consequence, we have:

Corollary 4.3 *If the characters $Q_m(\ell)$ of the representations $W_m(\ell)$ obey the recurrence relations in equations (4.2) and (4.3), then they must be given by the formula for combinatorial characters in Conjecture 2.6.*

The technique for proving Theorem 4.2 is as follows. The possible choices of initial data are limited by the requirement that $Q_1(\ell)$ be a representation with highest weight ω_ℓ . That is, $Q_1(\ell)$ must decompose into irreducible $U_q(\mathfrak{g})$ -modules as

$$Q_1(\ell) = V(\omega_\ell) + \sum_{\lambda \prec \omega_\ell} m_\lambda V(\lambda)$$

Note that we require that $V(\omega_\ell)$ occur in $Q_1(\ell)$ exactly once. Furthermore, we require that for every other component $V(\lambda)$ that appears, $\lambda \prec \omega_\ell$, *i.e.* that $\omega_\ell - \lambda$ is a positive root.

We proceed with a case-by-case proof. For each series, we find explicit multiplicities of irreducible representations occurring in $Q_m(\ell)$ which would be negative for any choice of $Q_1(\ell)$ other than that of Theorem 4.1. The calculations for series B , C and D are found below.

When \mathfrak{g} is of type A_n , no computations are necessary, because every fundamental root is minuscule: there are no $\lambda \prec \omega_\ell$ to worry about, no other choices for initial data to rule out. In fact, $Q_m(\ell)$ is just $V(m\omega_\ell)$ for all m and ℓ , and moreover every $U_q(\mathfrak{g})$ module is also acted upon by $U_q(\hat{\mathfrak{g}})$, by means of the evaluation representation.

B_n

Let \mathfrak{g} be of type B_n . Let V_i stand for $V(\omega_i)$ for $1 \leq i \leq n-1$, and V_{sp} for the character of the spin representation with highest weight ω_n . For convenience, let $\omega_0 = 0$ and V_0 denote the character of the trivial representation. Finally, we denote by V_n the character of the representation with highest weight $2\omega_n$, which behaves like the fundamental representations.

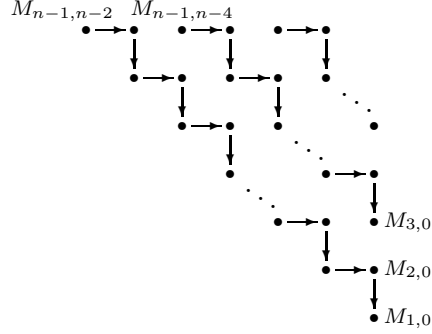
There are no dominant weights $\lambda \prec \omega_n$, so $Q_1(n) = V_{sp}$. The only weights $\lambda \prec \omega_a$ are $0, \omega_1, \dots, \omega_{a-1}$ for $1 \leq a \leq n-1$, so we write

$$Q_1(a) = V_a + \sum_{b=0}^{a-1} M_{a,b} V_b \tag{4.5}$$

Our goal is to prove that the only possible values for the multiplicities are

$$M_{a,b} = \begin{cases} 1, & a-b \text{ even} \\ 0, & a-b \text{ odd} \end{cases} \tag{4.6}$$

We will show these values are necessary inductively; the proof for each $M_{a,b}$ will assume the result for all $M_{c,d}$ with $\lceil \frac{c-d}{2} \rceil < \lceil \frac{a-b}{2} \rceil$ as well as those with $\lceil \frac{c-d}{2} \rceil = \lceil \frac{a-b}{2} \rceil$ and $c + d > a + b$. (Here $\lceil x \rceil$ is the least integer greater than or equal to x .) This amounts to working in the following order:



First follow the diagonal from $M_{n-1, n-2}$ to $M_{1,0}$, then the one from $M_{n-1, n-4}$ to $M_{3,0}$, etc., ending in the top right corner with $M_{n-1,0}$ or $M_{n-2,0}$, depending on the parity of n .

We show that equation (4.6) must hold for $M_{a,b}$, assuming it holds for all earlier M s in this ordering, by the following calculations:

1. For $M_{n-1,b}$ where $n-1-b$ is odd, the multiplicity of $V(\omega_b + \omega_n)$ in $Q_3(n)$ is $1 - 2M_{n-1,b}$,
2. For $M_{a,b}$ where $a-b$ is odd and $a \leq n-2$, the multiplicity of $V(\omega_{a+2} + \omega_b)$ in $Q_2(a+1)$ is $-M_{a,b}$,
3. For $M_{a,b}$ where $a-b$ is even:
 - The multiplicity of $V(\omega_a + \omega_b)$ in $Q_2(a)$ is $2M_{a,b} - 1$, and
 - The multiplicity of $V(\omega_{a+2} + \omega_b)$ in $Q_2(a+1)$ is $1 - M_{a,b}$.

Since all $M_{a,b}$ and all multiplicities are nonnegative integers, we must have $M_{a,b} = 0$ to satisfy the first two cases and $M_{a,b} = 1$ to satisfy the third.

The calculations to prove these claims depend on the ability to tensor together the $U_q(\mathfrak{g})$ -modules whose characters form $Q_m(\ell)$. A complete algorithm for decomposing these tensors is given in terms of crystal bases in [N]. For the current case, though, it happens that the only tensors we need to take are of fundamental representations. Simple explicit formulas for these decompositions had been given in [KN] before the advent of crystal base technology.

1. $M_{n-1,b}$, $n-1-b$ odd:

We want to find the multiplicity of $V(\omega_b + \omega_n)$ in $Q_3(n)$. Recursing through the polynomial relations, we find that

$$Q_3(n) = Q_1(n)^3 - 2Q_1(n)Q_1(n-1) = Q_1(n) [Q_1(n)^2 - 2Q_1(n-1)]$$

Assuming equation (4.6) for $M_{n-1,b'}$ for $b' > b$ and recalling that $Q_1(n)^2 = V_{sp}^2 = V_n + V_{n-1} + \cdots + V_0$, we need to compute the product

$$V_{sp} [V_n - V_{n-1} + V_{n-2} - \cdots - V_{b+1} + (1 - 2M_{n-1,b})V_b - \cdots]$$

Since $V_{sp}V_k = \sum_{i=0}^k V(\omega_i + \omega_n)$, we find that the multiplicity of $V(\omega_b + \omega_n)$ in the product is the desired $1 - 2M_{n-1,b}$.

2. $M_{a,b}$, $a - b$ odd, $a \leq n - 2$:

This calculation is typical of many of the ones that will follow, and will be written out in more detail. We want to know the multiplicity of $V(\omega_{a+2} + \omega_b)$ in $Q_2(a+1)$. When $a \leq n - 3$, we have

$$Q_2(a+1) = Q_1(a+1)^2 - Q_1(a+2)Q_1(a)$$

Assuming equation (4.6) holds for all earlier M s in the ordering, we have

$$\begin{aligned} Q_1(a+1) &= V_{a+1} + V_{a-1} + \cdots + V_b + M_{a+1,b-1}V_{b-1} + \cdots \\ Q_1(a+2) &= V_{a+2} + V_a + \cdots + V_{b+1} + M_{a+2,b}V_b + \cdots \\ Q_1(a) &= V_a + V_{a-2} + \cdots + V_{b+1} + M_{a,b}V_b + \cdots \end{aligned}$$

To compute $Q_1(a+1)^2 - Q_1(a+2)Q_1(a)$, we note that the V_sV_t term in $Q_1(a+1)^2$ and the $V_{s+1}V_{t-1}$ term in $Q_1(a+2)Q_1(a)$ are almost identical: when $s > t$, for example, the difference is just $\sum_{i=0}^t V(\omega_i + \omega_{s-t-2+i})$. In our case, the only $V(\omega_{a+2} + \omega_b)$ term that does not cancel out is the one contributed by $M_{a,b}V_{a+2}V_b$, and the multiplicity of $V(\omega_{a+2} + \omega_b)$ is $-M_{a,b}$.

When $a = n - 2$ the polynomial relations instead look like

$$Q_2(n-1) = Q_1(n-1)^2 - Q_1(n)^2Q_1(n-2) + Q_1(n-1)Q_1(n-2)$$

The $V_n + V_{n-2} + \cdots$ terms of $Q_1(n)^2$ behave just like the $Q_1(a+2)$ term in the above argument. The extra terms from $Q_1(n-2) [Q_1(n-1) - V_{n-1} - V_{n-3} - \cdots]$ make no net contribution, as can be seen by checking highest weights.

3. $M_{a,b}$, $a - b$ even:

Calculating the multiplicity of $V(\omega_{a+2} + \omega_b)$ in $Q_2(a + 1)$ is similar to the above; the trick of canceling $V_s V_t$ with $V_{s+1} V_{t-1}$ works again. The only terms remaining are $+1$ from $V_{a+1} V_{b+1}$ and the same $-M_{a,b}$ from $V_{a+2} M_{a,b} V_b$ as above, so the multiplicity is $1 - M_{a,b}$.

Likewise, calculating the multiplicity of $V(\omega_a + \omega_b)$ in $Q_2(a)$ we find two contributions of $M_{a,b}$ from $M_{a,b} V_a V_b$ (in either order) in $Q_2(a)^2$, and a contribution of 1 from $V_{a+1} V_{b+1}$ in $Q_1(a + 1) Q_1(a - 1)$, so the multiplicity is $2M_{a,b} - 1$.

 C_n

Let \mathfrak{g} be of type C_n . We let V_i stand for $V(\omega_i)$ for $1 \leq i \leq n$. The only dominant weights $\lambda \prec \omega_a$ for $1 \leq a \leq n$ are $\lambda = \omega_b$ for $0 \leq b < a$ and $a - b$ even, where $\omega_0 = 0$. (If $a - b$ is odd, then ω_a and ω_b lie in different translates of the root lattice, so are incomparable.) So we write

$$Q_1(a) = V_a + \sum_{i=0}^{\lfloor a/2 \rfloor} M_{a,a-2i} V_{a-2i} \quad (4.7)$$

We will prove that in fact $M_{a,b} = 0$ for all a and b .

Again we choose a convenient order to investigate the multiplicities: first look at $M_{a,a-2}$ for $a = n, n - 1, \dots, 2$, and then all $M_{a,b}$ with $a - b = 4, 6, 8, \dots$. This time the multiplicities acting as witnesses are:

1. For $M_{a,a-2}$, the multiplicity of $V(\omega_{a-1} + 2\omega_{a-2})$ in $Q_3(a - 1)$ is $1 - 2M_{a,a-2}$,
2. For $M_{a,b}$ for $a - b \geq 4$, the multiplicity of $V(\omega_{a-2} + \omega_b)$ in $Q_2(a - 1)$ is $-M_{a,b}$.

Performing these computations requires the ability to tensor more general representations of \mathfrak{g} than were needed in the B_n case. For this we use the generalization of the Littlewood-Richardson rule to all classical Lie algebras given in [N], which we summarize briefly in an Appendix to this chapter.

1. $M_{a,a-2}$:

We want to calculate the multiplicity of $V(\omega_{a-1} + 2\omega_{a-2})$ in $Q_3(a - 1)$. First we write $Q_3(a - 1)$ as a sum of terms of the form $Q_1(x)Q_1(y)Q_1(z)$, which we denote as $(x; y; z)$ for brevity. When $2 \leq a - 1 \leq n - 2$, we have

$$\begin{aligned} Q_3(a - 1) = & (a - 1; a - 1; a - 1) - 2(a; a - 1; a - 2) \\ & - (a + 1; a - 1; a - 3) + (a; a; a - 3) + (a + 1; a - 2; a - 2) \end{aligned}$$

When $a - 1$ is one of 1, 2 or $n - 1$, the above decomposition still holds, if we set $Q_1(0) = 1$ and $Q_1(-1) = Q_1(n + 1) = 0$. We want to find the multiplicity of $V(\omega_{a-1} + 2\omega_{a-2})$ in each of these terms.

First, $V(\omega_{a-1} + 2\omega_{a-2})$ occurs with multiplicity 3 in the V_{a-1}^3 component of $Q_1(a - 1)^3$. We calculate this number using the crystal basis technique for tensoring representations. Beginning with the Young diagram of V_{a-1} , we must choose a tableau $1, 2, \dots, a - 2, p$ from the second tensor factor, where p must be one of $a - 1$, a , or $\overline{a - 1}$. Then the choice of tableau from the third tensor component must be the same but replacing p with \overline{p} .

Similarly, the $V_a V_{a-1} V_{a-2}$ component of the $(a; a - 1; a - 2)$ term produces $V(\omega_{a-1} + 2\omega_{a-2})$ with multiplicity 1, corresponding to the choice of the tableau $1, 2, \dots, a - 2, \overline{a}$ from the crystal of V_{a-1} . We see that the remaining three terms cannot contribute by looking at tableaux in the same way.

Second, $V(\omega_{a-1} + 2\omega_{a-2})$ occurs in the $M_{a,a-2} V_{a-2} V_{a-1} V_{a-2}$ piece of $(a; a - 1; a - 2)$ and the $M_{a+1,a-1} V_{a-1} V_{a-2} V_{a-2}$ piece of $(a + 1; a - 2; a - 2)$ as the highest weight component. Our inductive hypothesis, however, assumes that $M_{a+1,a-1} = 0$, and we start the induction with $a = n$, where the $(a + 1; a - 2; a - 2)$ term vanishes entirely.

Totaling these results, we find that the net multiplicity is $1 - 2M_{a,a-2}$, and conclude that $M_{a,a-2} = 0$.

2. $M_{a,b}$ for $a - b \geq 4$:

We want to calculate the multiplicity of $V(\omega_{a-2} + \omega_b)$ in $Q_2(a - 1)$. For any $2 \leq a - 1 \leq n - 1$, we have

$$\begin{aligned} Q_2(a - 1) &= Q_1(a - 1)^2 - Q_1(a)Q_1(a - 2) \\ &= (V_{a-1} + \cdots)(V_{a-1} + \cdots) - (V_a + M_{a,b}V_b + \cdots)(V_{a-2} + \cdots) \end{aligned}$$

where every omitted term is either already known to be 0 by induction, or else has highest weight less than ω_b , so cannot contribute. As in the B_n case, the V_{a-1}^2 and $V_a V_{a-2}$ terms nearly cancel one another's contributions: their difference is just $\sum_{k=0}^{a-1} V(2\omega_k)$. Since $V(\omega_{a-2} + \omega_b)$ occurs in $V_{a-2}V_b$ with multiplicity 1, the net multiplicity in $Q_2(a - 1)$ is $-M_{a,b}$, and we conclude that $M_{a,b} = 0$.

D_n

Let \mathfrak{g} be of type D_n . This time we let V_i stand for $V(\omega_i)$ for $1 \leq i \leq n-2$, and use V_{n-1} for the character of the representation with highest weight $\omega_{n-1} + \omega_n$. We will not need to explicitly use the characters of the two spin representations individually, only their product, $V_{n-1} + V_{n-3} + \cdots$.

There are no dominant weights under ω_{n-1} or ω_n , and so no work to do on $Q_1(n-1)$ or $Q_1(n)$. For $1 \leq a \leq n-2$, the only dominant weights $\lambda \prec \omega_a$ are $\lambda = \omega_b$ for $0 \leq b < a$ and $a-b$ even; again $\omega_0 = 0$. (If $a-b$ is odd, then ω_a and ω_b lie in different translates of the root lattice, so are incomparable.) So we write

$$Q_1(a) = V_a + \sum_{i=0}^{\lfloor a/2 \rfloor} M_{a,a-2i} V_{a-2i} \quad (4.8)$$

We will show that in fact $M_{a,b} = 1$ for all a and b .

Again the proof is by induction; to show $M_{a,b} = 1$ we will assume $M_{c,d} = 1$ as long as either $c-d < a-b$ or $c-d = a-b$ and $c > a$. (This is the same ordering used for the B_n series after dropping the $M_{a,b}$ with $a-b$ odd.) Our witnesses this time are:

- The multiplicity of $V(2\omega_b)$ in $Q_2(a-1)$ is $1 - M_{a,b}$, and
- The multiplicity of $V(\omega_a + \omega_b)$ in $Q_2(a)$ is $2M_{a,b} - 1$.

We must therefore conclude that $M_{a,b} = 1$. Since we only need to tensor fundamental representations together, the explicit formulas given in [KN] are enough to carry out these calculations.

For any $\ell \leq n-3$, the polynomial relations give us

$$Q_2(\ell) = Q_1(\ell)^2 - Q_1(\ell+1)Q_1(\ell-1)$$

The multiplicity of $V(2\omega_b)$ in $Q_2(a-1)$ is easily calculated directly, since $V(2\omega_b)$ appears in $V_r V_s$ if and only if $r = s \geq b$, and then it appears with multiplicity one. The $Q_1(a-1)$ term therefore contains $V(2\omega_b)$ exactly $(a-b)/2$ times, while the $Q_1(a)Q_1(a-2)$ term subtracts off $M_{a,b} - 1 + (a-b)/2$ of them. Thus the net multiplicity is $1 - M_{a,b}$.

To calculate the multiplicity of $V(\omega_a + \omega_b)$ in $Q_2(a)$ for $a \leq n-3$, we once again use the trick of canceling the contribution from the $V_s V_t$ term of $Q_1(a)^2$ with the $V_{s+1} V_{t-1}$ term of $Q_1(a+1)Q_1(a-1)$. The cancellation requires more attention this time, since $V(\omega_a + \omega_b)$ occurs with multiplicity two in $V_s V_t$ when $a-b \geq 2n-r-s$.

In the end, the only terms that do not cancel are the contributions of $M_{a,b}$ from $V_a V_b$ and $V_b V_a$ in $Q_1(a)^2$ and of -1 from $V_{b+1} V_{a-1}$ in $Q_1(a+1)Q_1(a-1)$. Thus the net multiplicity is $2M_{a,b} - 1$.

Finally, if $a = n - 2$ the polynomial relations change to

$$Q_2(n-2) = Q_1(n-2)^2 - Q_1(n-1)Q_1(n)Q_1(n-3)$$

This change does not require any new work, though: $Q_1(n-1)Q_1(n)$ is just the product of the two spin representations, which decomposes as $V_{n-1} + V_{n-3} + \cdots$. Since this is exactly what we wanted $Q_1(\ell+1)$ to look like in the above argument, the preceding calculation still holds.

Appendix: Littlewood-Richardson Rule for C_n

This is a brief summary of a generalization of the Littlewood-Richardson rule to Lie algebras of type C_n , as given in [N]. For our purposes, we only need the ability to tensor an arbitrary representation with one of the fundamental representations with highest weights $\omega_1, \dots, \omega_n$.

The representation with highest weight $\sum_{k=1}^n a_k \omega_k$ is represented by a Young diagram Y with a_k columns of height k . For a fundamental representation V_k , we create Young tableaux from our column of height k by filling in the boxes with k distinct symbols i_1, \dots, i_k chosen in order from the sequence $1, 2, \dots, n, \bar{n}, \dots, \bar{2}, \bar{1}$ in all possible ways, as long as if $i_a = p$ and $i_b = \bar{p}$ then $a + (k - b + 1) \leq p$. These tableaux label the vertices of the crystal graph of the representation V_k .

Given a Young diagram Y , the symbols $1, 2, \dots, n$ act on it by adding one box to the first, second, \dots , n th row, and the symbols $\bar{1}, \bar{2}, \dots, \bar{n}$ act by removing one, provided the addition or removal results in a diagram whose rows are still nonincreasing in length. The result of the action of the symbol i_a on Y is denoted $Y \leftarrow i_a$.

Then the tensor product $V \otimes V_k$, where V has Young diagram Y , decomposes as the sum of all representations with diagrams $((Y \leftarrow i_1) \leftarrow i_2) \cdots \leftarrow i_k$, where i_1, \dots, i_k range over all tableaux of V_k such that each of the actions still result in a diagram whose rows are still nonincreasing in length.

Chapter 5

Summary and Further Questions

Our goal was to study finite-dimensional representations of Yangians and quantum affine algebras. We began with a conjectural formula of Kirillov and Reshetikhin's, based on methods of mathematical physics, describing how some of these representations should decompose into irreducible representations of the underlying finite-dimensional Lie algebras.

In Chapter 3, we gave a new combinatorial interpretation of this formula. This new point of view made possible computations which were completely intractable using the original version. This formulation also endows the decomposition into irreducibles with a new tree structure, if the underlying Lie algebra is simply-laced. We used this tree to note some structural compatibility among representations whose highest weights were different multiples of the same fundamental weight, and were able to calculate the asymptotics of the growth of their dimension.

In Chapter 4, we explored a set of polynomial relations that seem to hold among the characters of some of these representations. The fact that every finite-dimensional representation of the quantum affine algebra is a direct sum of representations of the underlying Lie algebra is a very strong positivity condition. We proved that for the classical families of Lie algebras, the positivity condition and the polynomial relations leave only one choice for the characters of the quantum affine algebras.

We conclude with some questions for further research which seem to come naturally from the topics discussed here.

1. First and foremost, the conjectural Kirillov-Reshetikhin formula requires a solid mathematical proof. In light of our results in Chapter 4, it would suffice to prove that the characters of the representations $W_m(\ell)$ (as \mathfrak{g} -modules)

actually do satisfy the polynomial relations. One possible way to prove this would be by showing the representations form an exact sequence, something like

$$0 \rightarrow W_{m+1}(\ell) \otimes W_{m-1}(\ell) \rightarrow W_m(\ell) \otimes W_m(\ell) \rightarrow \bigotimes_{\ell' \sim \ell} \mathcal{W}(m, \ell, \ell') \rightarrow 0$$

If we want to treat these as tensor products of $U_q(\hat{\mathfrak{g}})$ -modules, we need to specify shifts as well. One can hope that the correct shifts are given by the “T-system” written down in [KNS], a proposed generalization of these polynomial relations to include shifts. It is also possible that there are two different choices for the shifts such that the above exact sequence is correct for one and backwards for the other.

2. It seems reasonable to hope that these results apply to cases other than just $U_q(\hat{\mathfrak{g}})$. Certainly the twisted simply-laced affine cases are the most natural candidates. One can also hope that the polynomial relations hold if we replace \mathfrak{g} with a more general Kac-Moody algebra. Of course, the easy definition of $U_q(\hat{\mathfrak{g}})$ is no longer available. But since the results of Chapter 4 do not rely on knowing what representations of $U_q(\hat{\mathfrak{g}})$ look like, it may be possible to generalize the polynomial relations and find a unique solution first, and only later identify the solutions as irreducible representations of some new object.
3. The tree structure introduced here is, so far, only a little better than a computational tool. The fact that it highlighted a similarity in structure between representations with different highest weights, though, indicates that there may be some representation theoretic meaning to the way it arranges pieces in the decomposition. If so, there might be a similar structure to decompositions for non-simply-laced cases, even though this construction does not make sense.

A probable first step would be to understand better the structure in the A_n case, where a body of knowledge exists on tensor products of rectangles. Discretion is required, though: any patterns noticed for A_n can generalize either to the other classical finite-dimensional Lie algebras or to representations of Yangians.

4. The original Kirillov-Reshetikhin formula, or even just the tree structure in the A_n case, should have a generalization to representations with highest weights other than just multiples of a fundamental weight. Unfortunately, the algorithm for tensoring rectangles, which is phrased in a way that would make

sense for tensoring arbitrary shapes, is certainly not true beyond the rectangle case: as presented here, it only depends on the multisets of n th columns of its arguments, which the tensor product does not.

There is another potential stumbling block in generalizing these results in cases other than A_n : it is no longer clear which $U_q(\mathfrak{g})$ -modules we should be decomposing. The modules $W_m(\ell)$ studied here were defined by assigning a canonical choice of Drinfeld polynomials to any weight of the form $m\omega_\ell$. It is unclear how this assignment should be generalized to all weights, or even if there is a correct generalization.

Chari and Pressley have investigated the notion of a “minimal affinization” of a $U_q(\mathfrak{g})$ -module at some length, and report that while there is indeed a canonical representation of $U_q(\hat{\mathfrak{g}})$ in our cases, often there is not. For a generic highest weight of D_n and E_n , in fact, there are three! While the modules $W_m(\ell)$ coincided with these “minimal affinizations” so far, there is no guarantee that the correct generalization would continue to do so.

5. Finally, the polynomial relations among the characters should also be generalized to highest weights other than multiples of a fundamental weight. Again, this would be interesting even in the A_n case. Preliminary investigation in this direction supports the guess that there is a generalization of the form

$$V(\lambda + \omega_\ell)^2 = V(\lambda)V(\lambda + 2\omega_\ell) + \sum_k V(\mu_{i_k})V(\mu_{j_k})$$

where the weights μ in the sum of products are not very far away from λ and should have height at most that of $\lambda + \omega_\ell$. This again has the feeling of a discrete dynamical system about it. If one were to generalize the results of Chapter 4 to such a system of relations, the set of “base cases” would grow at least to $2^{\text{rank}(\mathfrak{g})} - 1$, since the recurrence relation would only give you values for highest weights of the form $\lambda + 2\omega_\ell$. Again, the fact that there is not a canonical representation of $U_q(\mathfrak{g})$ with a given highest weight makes things trickier.

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